Lecture 26

Last time, developed partial wave expansion

Scattering wave function in general:

$$\psi \to A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right] \quad \text{for large } r$$

Reduced to $\psi \to A\left[e^{ikz} + k\sum_\ell (2\ell+1)\, a_\ell\, P_\ell(\cos\theta)\, i^{\ell+1} h_\ell^{(1)}(kr)\right]$

for $r >$ range of V

ℓ = angular momentum, "partial wave"
a_ℓ = partial wave amplitude ($\neq$ scattering length)
P_ℓ = Legendre polynomial
$h_\ell^{(1)}$ = spherical Hankel function

Need to get a_ℓ's

Last piece: write e^{ikz} as partial wave

Kind of tedious, see supplement on web

Get $e^{ikz} = \sum_\ell i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$

Then we have

$$\psi \to A\sum_\ell i^\ell (2\ell+1)\left[j_\ell(ka) + ika_\ell\, h_\ell^{(1)}(ka)\right] P_\ell(\cos\theta)$$

This is already enough to solve the hard sphere problem

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

And ψ has free-particle form for $r > a$

Also have boundary condition $\psi = 0$ at $r = a$

So $\sum_\ell i^\ell (2\ell+1)\left[j_\ell(ka) + ika_\ell h_\ell^{(1)}(ka)\right] P_\ell(\cos\theta) = 0$

But P_ℓ's are independent functions, so coefficient must vanish for each ℓ:

$$j_\ell(ka) + ika_\ell h_\ell^{(1)}(ka) = 0$$

$$\boxed{a_\ell = \frac{i}{k} \frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}}$$

From that, get $D(\theta)$, σ etc. $f(\theta) = \sum (2\ell+1) a_\ell P_\ell(\cos\theta)$

But kind of obscure. Look at low energy limit $k \to 0$

Have $\frac{j_\ell(x)}{h_\ell^{(1)}(x)} = \frac{j_\ell(x)}{j_\ell(x) + in_\ell(x)}$

From table or pg 143: for small x,

$$j_\ell(x) \to \frac{2^\ell \ell!}{(2\ell+1)!} x^\ell \to 0 \quad \text{for } \ell > 0$$

$$n_\ell(x) \to -\frac{(2\ell)!}{2^\ell \ell!} \frac{1}{x^{\ell+1}} \to \infty$$

So $\frac{j_\ell}{h_\ell^{(1)}} \to \frac{j_\ell}{in_\ell} \to \frac{1}{i}\left[\frac{2^\ell \ell!}{(2\ell+1)!} x^\ell\right] \cdot \left[-\frac{2^\ell \ell!}{(2\ell)!} x^{\ell+1}\right]$

$$\frac{i}{2\ell+1}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 x^{2\ell+1}$$

and $\quad a_\ell \to -\frac{1}{k}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 (ka)^{2\ell+1}$

Specifically:

$$a_0 = -a$$

$$a_1 = -k^2 a^3$$

$$a_2 = -\frac{1}{9} k^4 a^5$$

Note $\quad a_0 \propto (E)^0$

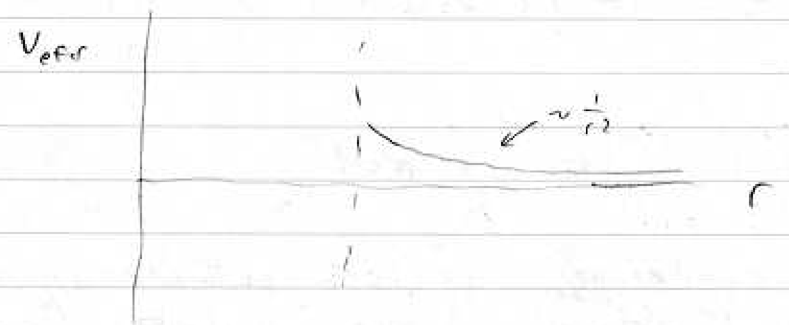
$a_1 \propto E$

$a_2 \propto E^2 \qquad$ at low energy, s-wave scattering dominates

This is easy to understand:

Effective potential is $V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}$

If V has finite range, then ang. mom. term dominates for large r



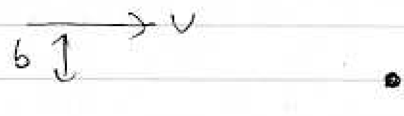
If E is low enough, particle is repelled by ang. mom. barrier before it ever penetrates in far enough to see V

If it doesn't see V, then it behaves like a free particle ... no scattering at all $(a=0)$

True for all but $l=0$ ... no barrier there.

Another way to see it, more classically
If incident particle has $l>0$, it
isn't headed straight at scatterer

$L = mvb$



For fixed L, as v decreases, b must
increase.
Eventually b exceeds range of potential and
scattering stops.

So as $E \to 0$, just have $a_0 = -a$

Then $\sigma = 4\pi \sum_l (2l+1)|a_l|^2 \to 4\pi a^2$
= surface area of sphere

Recall classical limit $\sigma = \pi a^2$

In s-wave scattering, incident particle comes
from all directions ... sees whole surface.