W: 11.8
F: 11.10

Lecture 27

Last, finished development of partial wave expansion

Write scattering wave function

$$\psi \to A \sum_\ell i^\ell (2\ell+1)\left[j_\ell(kr) + ika_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta)$$

Solved hard sphere scattering

$$a_\ell = \frac{i}{k} \frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}$$

Saw that for low incident energies, low ℓ terms dominate

At any energy, only finite # of terms needed

But getting a_ℓ's in general not so easy

Method:

Start by solving radial Sch Eqn for given k, ℓ
- Not so hard, can do numerically

Use boundary condition $u(r) \to 0$ at $r=0$, like usual

Don't worry about scattering problem, just get $u(r)$

Know that for large r, $\psi \sim$ free particle

So $R_\ell(r) \to C j_\ell(kr) + D n_\ell(kr)$

C & D depend on details of potential

But at really large r,

$$j_\ell(kr) \to \frac{\sin(kr - \ell\pi/2)}{kr}$$

$$n_\ell(kr) \to -\frac{\cos(kr - \ell\pi/2)}{kr}$$

So $R_\ell(r) \to \frac{1}{kr}\left[C\sin(kr - \ell\frac{\pi}{2}) - D\cos(kr - \ell\frac{\pi}{2})\right]$

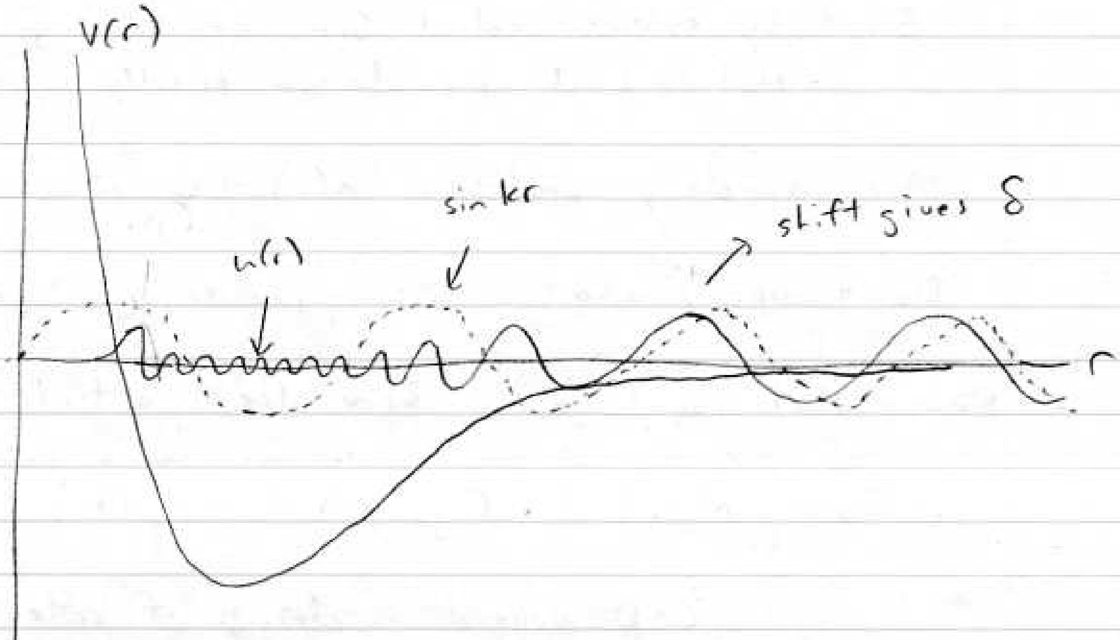
$$= \frac{1}{kr} B\sin(kr - \ell\tfrac{\pi}{2} + \delta_\ell)$$

for some B & δ_ℓ

Note that δ_ℓ is pretty easy to get from your computed $u(r)$:

Just compare $u(r)$ to $\sin(kr - \ell\frac{\pi}{2})$
& note phase shift

For instance: $(\ell = 0)$



So if we can solve Schr eqn, can get δ

But how is that related to scattering?

Decompose solution into incoming + outgoing waves

$$\psi_\ell \to B_\ell \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} P_\ell(\cos\theta)$$

$$= \frac{B_\ell}{kr}\frac{1}{2i}\left[e^{i(kr-\ell\pi/2+\delta_\ell)} - e^{-i(kr-\ell\pi/2+\delta_\ell)}\right] P_\ell(\cos\theta)$$

$$= \frac{B_\ell}{kr}\frac{1}{2i}\left[i^{-\ell} e^{i(kr+\delta_\ell)} - i^{\ell} e^{-i(kr+\delta_\ell)}\right] P_\ell(\cos\theta)$$

$$\boxed{= \frac{B_\ell}{2ikr} i^{-\ell} e^{-i\delta_\ell}\left[e^{2i\delta_\ell} e^{ikr} - i^{2\ell} e^{-ikr}\right] P_\ell(\cos\theta)}$$

Compare this to other form:

$$\psi_\ell \to A i^\ell (2\ell+1)\left[\frac{\sin(kr - \ell\frac{\pi}{2})}{kr} + ika_\ell (-i)^{\ell+1} \frac{e^{ikr}}{kr}\right] P_\ell(\cos\theta)$$

$$= \frac{A}{kr} i^\ell (2\ell+1)\left[\frac{1}{2i}\left(e^{i(kr-\ell\frac{\pi}{2})} - e^{-i(kr-\ell\frac{\pi}{2})}\right)\right.$$

$$\left. + ka_\ell (-i)^\ell e^{ikr}\right] P_\ell(\cos\theta)$$

$$= \frac{A}{2kr}(2\ell+1) i^\ell \left[i^{-(\ell+1)} e^{ikr} - i^{(\ell-1)} e^{-ikr} + 2ka_\ell i^{-\ell} e^{ikr}\right]$$

$$\times P_\ell(\cos\theta)$$

$$= \frac{A}{2kr}(2\ell+1)\left[\left(\frac{1}{i} + 2ka_\ell\right) e^{ikr} - i^{(2\ell-1)} e^{-ikr}\right] P_\ell(\cos\theta)$$

$$\boxed{= \frac{A}{2ikr}(2\ell+1)\left[(1 + 2ika_\ell) e^{ikr} - i^{2\ell} e^{-ikr}\right] P_\ell(\cos\theta)}$$

Identifying terms, see

$$B_\ell\, i^{-\ell} e^{-i\delta_\ell} = A(2\ell+1)$$ normalization, not too important

and $$\boxed{e^{2i\delta_\ell} = 1 + 2ika_\ell}$$

Thus $$a_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) = \boxed{\frac{1}{k} e^{i\delta_\ell} \sin\delta_\ell}$$

So if we calculate δ_ℓ, easy to get a_ℓ
→ scattering problem solved
(given solution to radial Sch. Eqn)

Phase shift has nice physical interpretation
(cf picture)

Think about low energy limit

Generally $a_\ell \to \beta k^{2\ell}$ as $k \to 0$

So $e^{i\delta_\ell} \sin\delta_\ell \to \beta k^{2\ell+1}$
$\delta_\ell \to 0$ for all ℓ

Since δ_ℓ is small, $\delta_\ell \approx \beta k^{2\ell+1}$

Define s-wave scattering length

$$\alpha = -\lim_{k\to 0} \frac{\delta_0}{k} = -a_0$$

For hard sphere, $\alpha = a$ = radius of sphere