Lecture 28

Last time; finished partial wave expansion.

General method:
1) For given $k, l$, solve radial Sch. Eqn. for $u_l(r)$
2) At large $r$, $u_l(r) \Rightarrow B \sin (kr - l \pi/2 + \delta) / r^l$; set $\delta$
3) Get scattering amplitude $a_l = \frac{i}{2k} e^{-i \delta} \sin \delta$
4) Differential cross section
   \[ \frac{d}{d\theta} = \frac{2}{\pi} \frac{1}{l+1/2} \frac{a_l^* a_l}{r^l} P_l(\cos \theta) P_{l+1}(\cos \theta) \]

Guaranteed that $a_l \to 0$ for large enough $l$ (at fixed $k$)

This always works, but it is kind of a project, especially since you typically have variable $k$

More convenient method: Born approximation

To develop this, start by recasting Sch. Eqn. as an integral equation instead.

What's an integral equation?
Like differential equation, but involving integrals instead of derivatives.

Easiest to see example
Start with $-\frac{\mathbf{\nabla}^2}{2m} \mathbf{\nabla}^2 + V \mathbf{\nabla}^2 = E \mathbf{\nabla}^2$

For convenience, rewrite as

$\mathbf{\nabla}^2 \mathbf{\nabla}^2 + \frac{2mE}{\hbar^2} \mathbf{\nabla}^2 = \frac{2mE}{\hbar^2} V \mathbf{\nabla}^2$

$(\mathbf{\nabla}^2 + k^2)^2 = \infty$

$k = \sqrt{\frac{2mE}{\hbar^2}} \implies Q = \frac{2mE}{\hbar^2} \mathbf{\nabla}^2$

$\Rightarrow$ in scattering problems, $k$ is given, $V(r) =$ "source" of scattered wave

Suppose we solved related problem:

$(\mathbf{\nabla}^2 + k^1)^2 G(r) = \delta^3(r)$

Then we could express $\mathbf{\nabla}^2$ as

$\mathbf{\nabla}^2 = \int G(r-r_0) \mathbf{\nabla} \delta(r_0) d^3 r$

Check:

$(\mathbf{\nabla}^2 + k^1)^2 = \int \left[ (\mathbf{\nabla}^2 + k^1) G(r-r_0) \right] \mathbf{\nabla} (r_0) d^3 r_0$

$= \int \left[ \delta^3(r-r_0) \right] \delta(r_0) d^3 r_0$

$= \delta(r)$ as needed.

This is a standard trick for solving inhomogeneous equations.

$G(r)$ called "Green's function".

Generally, Green's function = solution to diff eq. with 8-ken source term.
But here we're doing something a little different:
Because $Q$ has $2\pi$ in it.

So really,

$$2\pi(\vec{r}) = \frac{2\pi}{\delta^{2}} \int \delta(\vec{r} - \vec{r}_0) \nabla \phi(\vec{r}_0) \, d^3 r_0$$

not a true solution: can't evaluate integral
unless we know $2\pi$.

This is an integral equation.

See why it's useful next time.
For now, want to get $G(\vec{r})$.

Two ways to solve:
"Easy" way: guess solution and plug into check
Matt will show us that.

"Hard" way: derive using Fourier transform & contour integration
You are big kids now, I'll show you the hard way.

Fourier transform:

Have $G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{r} \cdot \vec{s}} g(\vec{s}) \, d^3 s$

with $g(\vec{s}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{s} \cdot \vec{r}} G(\vec{r}) \, d^3 r$

(transform pair)
Apply $\triangle^2 + k^2$ to $G(\vec{r})$:

$$\triangle^2 + k^2 G(\vec{r}) = \frac{1}{(2\pi)^3} \int \left[ (\triangle^2 + k^2) e^{i\vec{k} \cdot \vec{r}} \right] g(\vec{s}) \, d^3 \vec{s}$$

$$= \frac{1}{(2\pi)^3} \int \left[ -\vec{s}^2 + k^2 \right] e^{i\vec{s} \cdot \vec{r}} g(\vec{s}) \, d^3 \vec{s}$$

$$= \delta^3(\vec{r})$$

But also know $\delta^3(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{s} \cdot \vec{r}} \, d^3 \vec{s}$

Combine integrals:

$$\int \left[ \frac{1}{(2\pi)^3} - \frac{g(\vec{s})}{(2\pi)^3} (k^2-s^2) \right] e^{i\vec{s} \cdot \vec{r}} \, d^3 \vec{s} = 0 \quad \text{for all } \vec{r}$$

Term in $[ ]$'s must be zero, so

$$g(\vec{s}) = \frac{1}{(2\pi)^3} \frac{1}{k^2-s^2}$$

That's the Fourier transform of $G$.

So $G(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k} \cdot \vec{r}} \frac{1}{k^2-s^2} \, d^3 \vec{s}$

See how to solve this next time.
Note:

Proof that \( \nabla^2 r = -4\pi \delta^3(r) \)

In spherical coordinates, \( \nabla^2 r = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \) angular derivatives.

So \( \nabla^2 r = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \left( -\frac{1}{r^2} \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left( -1 \right) = 0 \)

\( \nabla^2 r = 0 \) for \( r > 0 \), but is indefinite when \( r = 0 \) (not \( \frac{\partial}{\partial r} \)).

To see what's going on at \( r = 0 \), use Gauss's Theorem:

For any \( F(x) \), \( \iiint_{\mathbb{R}^3} \mathbf{F} \cdot d\mathbf{A} = \iiint_{\mathbb{R}^3} \mathbf{F} \cdot d\mathbf{A} \)

Take \( F = \nabla r = \left( \frac{x}{x^2 + y^2 + z^2} \right) \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \)

\( = \frac{x}{x^2 + y^2 + z^2} - \frac{1}{r^2} \)

\( = -\frac{1}{r^2} \)

Consider \( U = \) sphere of radius \( R \) centered at origin.

Then \( -\iiint_{\mathbb{R}^3} \mathbf{F} \cdot d\mathbf{A} = -\frac{1}{r^2} \iiint_{\mathbb{R}^3} d\mathbf{A} = -\frac{1}{r^2} 4\pi R^2 = -4\pi \)

But also, \( \nabla \cdot F = \nabla^2 r \)

So, \( \iiint_{\mathbb{R}^3} \nabla^2 r = -4\pi \), no matter how small \( r \) is.

That means \( \nabla^2 r \) must act like \( \delta^3(r) \).

Specifically, \( \nabla^2 r = -4\pi \delta^3(r) \).