Lecture 28

Last time; finished partial wave expansion.

General method:

1) For given k, ℓ, solve radial Sch. Eqn for $u_\ell(r)$

2) At large r, $u_\ell(r) \to B\sin(kr - \ell\frac{\pi}{2} + \delta_\ell)$: get δ_ℓ

3) Get scattering amplitude $a_\ell = \frac{1}{2k} e^{i\delta_\ell} \sin\delta_\ell$

4) Differential cross section
$$D(\theta) = \sum_{\ell\ell'} (2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'}\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

Guaranteed that $a_\ell \to 0$ for large enough ℓ (at fixed k)

This always works, but it is kind of a project
- especially since you typically have variable k

More convenient method: Born approximation

To develop this, start by recasting Sch. Eqn as an <u>integral</u> <u>equation</u> instead.

What's an integral equation?
Like differential equation, but involving integrals instead of derivatives.

Easiest to see example

Start with $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$

For convenience, rewrite as

$$\nabla^2\psi + \frac{2mE}{\hbar^2}\psi = \frac{2m}{\hbar^2}V\psi$$

$$(\nabla^2 + k^2)\psi = Q$$

$$k = \sqrt{\frac{2mE}{\hbar}} \qquad Q = \frac{2mV}{\hbar^2}\psi$$

$\Rightarrow$ in scattering problems, k is given,
$V(r)$ = "source" of scattered wave

Suppose we solved related problem:

$$(\nabla^2 + k^2)\,G(\vec{r}) = \delta^3(\vec{r})$$

Then we could express ψ as

$$\psi(\vec{r}) = \int G(\vec{r}-\vec{r}_0)\,Q(\vec{r}_0)\,d^3r$$

Check:

$$(\nabla^2+k^2)\psi = \int\left[(\nabla^2+k^2)\,G(\vec{r}-\vec{r}_0)\right]Q(\vec{r}_0)\,d^3r_0$$

$$= \int\left[\delta^3(\vec{r}-\vec{r}_0)\right]Q(\vec{r}_0)\,d^3r_0$$

$$= Q(\vec{r}) \quad \text{as needed.}$$

This is a standard trick for solving inhomogeneous equations.

$G(\vec{r})$ called "Green's function"

Generally, Green's function = solution to diff eq. with δ-fcn source term.

But here we're doing something a little different:
Because Q has ψ in it

So really,

$$\psi(\vec{r}) = \frac{2m}{\hbar^2} \int G(\vec{r}-\vec{r}_0)\, V(\vec{r}_0)\, \psi(\vec{r}_0)\, d^3 r_0$$

not a true solution: can't evaluate integral unless we know ψ.

This is an integral equation.

See why its useful next time.
For now, want to get $G(\vec{r})$

Two ways to solve:
"Easy" way: guess solution and plug into check
Matt will show us that.

"Hard" way: derive using Fourier transform & contour integration

You are big kids now, I'll show you the hard way.

Fourier transform:

$$\text{Have } G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{s}\cdot\vec{r}}\, g(\vec{s})\, d^3\vec{s} \qquad \Big| \text{ transform pair}$$

$$\text{with } g(\vec{s}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{s}\cdot\vec{r}}\, G(\vec{r})\, d^3\vec{r}$$

Apply $\nabla^2 + k^2$ to $G(\vec{r})$:

$$(\nabla^2 + k^2)\, G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int \left[(\nabla^2 + k^2)\, e^{i\vec{s}\cdot\vec{r}}\right] g(\vec{s})\, d^3\vec{s}$$

$$= \frac{1}{(2\pi)^{3/2}} \int (-s^2 + k^2)\, e^{i\vec{s}\cdot\vec{r}}\, g(\vec{s})\, d^3\vec{s}$$

$$= \delta^3(\vec{r})$$

But also know $\delta^3(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}}\, d^3\vec{s}$

Combine integrals:

$$\int \left[\frac{1}{(2\pi)^3} - \frac{g(\vec{s})}{(2\pi)^{3/2}} (k^2 - s^2)\right] e^{i\vec{s}\cdot\vec{r}}\, d^3\vec{s} = 0 \quad \text{for all } \vec{r}$$

Term in [ ]'s must be zero, so

$$\boxed{g(\vec{s}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{k^2 - s^2}}$$

That's the Fourier transform of G.

So $\boxed{G(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{s}\cdot\vec{r}} \frac{1}{k^2 - s^2}\, d^3 s}$

See how to solve this next time

Note:

Proof that $\nabla^2 \frac{1}{r} = -4\pi\, \delta^3(\vec{r})$

In spherical coords, $\nabla^2 \to \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$ + angular derivatives

So $\nabla^2 \frac{1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \cdot \left(-\frac{1}{r^2}\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0$

$\nabla^2 \frac{1}{r} = 0$ for $r > 0$, but is indefinite when $r = 0$ (get $\frac{0}{0}$)

To see what's going on at $r = 0$, use Gauss's Theorem:

For any $\vec{F}(\vec{r})$, $\iiint \vec{\nabla}\cdot\vec{F}\, dV = \oint\!\!\oint \vec{F}\cdot d\vec{A}$

Take $\vec{F} = \vec{\nabla}\frac{1}{r} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$= -\frac{\hat{x}x + \hat{y}y + \hat{z}z}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$$

Consider V = sphere of radius R centered on origin.

Then $-\oint\!\!\oint \frac{\hat{r}}{r^2}\cdot d\vec{A} = -\frac{1}{R^2}\iint dA = -\frac{1}{R^2}\cdot 4\pi R^2 = -4\pi$

But also, $\vec{\nabla}\cdot\vec{F} = \nabla^2 \frac{1}{r}$

So $\iiint \nabla^2 \frac{1}{r}\, dV = -4\pi$, no matter how small R is

That means $\nabla^2 \frac{1}{r}$ must act like $\delta^3(\vec{r})$

Specifically, $\boxed{\nabla^2 \frac{1}{r} = -4\pi\, \delta^3(\vec{r})}$