Lecture 29

Last time, started on Born Approx.

Integral version of Sch. Eqn:

\[ \psi(r) = \frac{2m}{\hbar^2} \int G(r-r_0) \psi(r_0) \, \text{d}^3r_0 \]

where \( G(r) \) is Green's function, satisfies

\[ (\nabla^2 + k^2) G = \delta^3(r) \]

Last time, worked out

\[ G(r) = \frac{1}{(2\pi)^3} \int e^{-i \mathbf{k} \cdot \mathbf{s}'} \, \text{d}^3s' \]

\[ = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} e^{i\mathbf{k} \cdot \mathbf{s}} \, \sin \theta \, \cos \phi \, \text{d} \Omega \]

\[ = \frac{1}{(2\pi)^3} \int_0^\pi \int_0^{2\pi} \sin^2 \theta \, \sin \phi \, \cos \phi \, \text{d} \phi \, \text{d} \theta \]

\[ = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^\pi \sin \theta \, \sin \phi \, \cos \phi \, \text{d} \theta \, \text{d} \phi \]

\[ = \frac{1}{(2\pi)^3} \int_0^{2\pi} \frac{1}{2} \int_0^\pi 2 \sin \phi \, \cos \phi \, \text{d} \phi \, \text{d} \theta \]

Note integral is even in \( \phi \).

Can write

\[ = \frac{1}{(2\pi)^3} \frac{1}{2} \int_0^{2\pi} \int_0^\pi \frac{\sin \theta}{\mathbf{k}^2 + \mathbf{s}^2} \, \text{d} \theta \, \text{d} \phi \]

But actually, go back to exp form now...
\[ G = \frac{1}{2\pi i} \left[ \int_{C} \frac{se^{isr}}{k^2-s^2} \, ds - \int_{C} \frac{se^{isr}}{k^2-s^2} \, ds \right] \]

Do these with contour integration.

Generalization of integrals to complex plane:

\[ \int f(z) \, dz \text{ for } z \text{ complex} \]

Now need to specify path in complex plane:

\[ \text{Much like a 2D line integral} \]

But not exactly the same. If \( \frac{df}{dz} \) is to be well-defined, need some relationship between \( \text{Re} z \) and \( \text{Im} z \).

Most important result: Cauchy's formula:

\[ \oint_{\gamma} \frac{f(z)}{z-z_0} \, dz = \begin{cases} 0 & \text{if } z_0 \text{ outside loop} \\ 2\pi i f(z_0) & \text{if } z_0 \text{ inside loop} \end{cases} \]

Closed integral = contour (Assuming \( f(z) \) itself has no discontinuities inside the contour).

Cell \( z_0 \) a pole = place where integral \( \to \infty \) like \( \frac{1}{z} \)

* Need to traverse contour in counter-clockwise sense.
Otherwise, get - sign.
How do we apply it here?

One problem: integration path isn't a closed loop.

That's OK, we can close it at \( \infty \), as long as integrand goes to zero for large \( \|z\| \).

For \( \int \frac{se^{-is}}{k|z|^s} \, ds \), close with \( \text{Im} s = 0 \): Then \( e^{-ix} \) goes to \( 1 \) for large \( \|z\| \).

For \( \int \frac{se^{-is}}{k|z|^s} \, ds \), need to close with \( \text{Im} s < 0 \).

In either case, can show that \( \int ds \to 0 \) since integrand gets exponentially small.
Second problem: \( \frac{1}{k^2 - s^2} = \frac{1}{(k-s)(k+s)} \)

Poles at \( s = \pm k \)

Neither inside nor outside contour! Cauchy's Theorem doesn't apply.

Makes sense: \( \int_0^\infty \frac{e^{-ks}}{k-s} \, ds = \infty \), diverges like \( \frac{1}{s} \)

But really, total integral is finite:

\[
G(r) = \frac{1}{4\pi^2} \int_0^{\infty} \frac{S \sin \theta r}{(k-s)(k+s)} \, ds
\]

\[
= B + \frac{1}{2\pi^2} \int_0^{\infty} \frac{S \sin \theta r}{k-s} \, ds + \frac{1}{2\pi^2} \int_{-k}^{-\infty} \frac{S \sin \theta r}{k-s} \, ds
\]

Something finite.

\[
= B + \frac{1}{2\pi^2} \left[ \frac{k \sin \theta r}{2k} \int_{-k}^{0} \frac{1}{k-s} \, ds + \frac{(-k) \sin \theta r}{72k} \int_{-k}^{-\infty} \frac{1}{k-s} \, ds \right]
\]

\[
= B + \frac{1}{2\pi^2} \left[ -\frac{\sin \theta r}{2k} \int_{-k}^{0} \frac{1}{s-k} \, ds + \frac{\sin \theta r}{2k} \int_{-k}^{-\infty} \frac{1}{s+k} \, ds \right]
\]

\[
= B + \frac{1}{2\pi^2} \left[ -\left( \int_{-k}^{0} \frac{1}{s-k} \, ds + \int_{-\infty}^{-k} \frac{1}{s+k} \, ds \right) \right]
\]

\[= 0 \text{ for any } \epsilon \]

So divergent parts cancel out, get finite answer.

That means that in contour integral, what happens right near the poles is not so important.
Instead of

Lose

Use

Easiest to include just one pole

Infinities in "bump" around pole

Could deform contour differently... still get a valid solution:

If $G'(z) = G(z) + H(z)$ with $(\Delta^2 + k^2)H = 0$, then $G'$ is a valid Green's func.

Shifting contour near poles just changes $H$

Choice shown is simplest.

So we can finally apply Cauchy's formula

$$
\int_{\gamma} \frac{se^{isr}}{k^2 - s^2} \, ds = -\delta \left( \frac{se^{isr}}{s + k} \right) \frac{1}{s - k} \, ds
$$

$$
= -2\pi i \frac{ke^{ikr}}{2k}
$$

$$
= -\pi i e^{ikr}
$$
For other integral, need to use some deformation

Note, contour has clockwise sense

Then
\[
\int_{-\infty}^{\infty} \frac{se^{-is\tau}}{s^2 + 2\sigma^2 s + \sigma^4} \, ds = -\frac{2\pi i}{2\sigma^2} \int_{\gamma} \frac{se^{-is\tau}}{s+1} \, ds
\]

\[
= +2\pi i \left( \frac{(-\tau)i\sigma}{1-2\sigma^2} \right) = 4\pi i \tau e^{-\sigma^2/2}
\]

And have

\[
G(1) = \frac{1}{2\pi i} \left[ \int_0^\infty \frac{se^{-is\tau}}{s^2 + 2\sigma^2 s + \sigma^4} \, ds - \int_{\gamma} \frac{se^{-is\tau}}{s^2 + 2\sigma^2 s + \sigma^4} \, ds \right]
\]

\[
= \frac{1}{4\pi i} \left[ -i\pi e^{-\sigma^2/2} - i\pi e^{-\sigma^2/2} \right] = -i\pi e^{-\sigma^2/2}
\]

\[
G(2) = -\frac{i}{4\pi e^{-\sigma^2/2}}
\]

As Matt showed us last time, this is correct answer.