Lecture 30

So far:

Integral form of Schr Eqn:

$$\psi(\vec{r}) = \frac{2m}{\hbar^2} \int G(\vec{r}-\vec{r}_0)\, V(\vec{r}_0)\, \psi(\vec{r}_0)\, d^3r_0$$

$V(\vec{r})$ = potential

$G(\vec{r})$ = Green's function $= -\frac{1}{4\pi r} e^{ikr}$

Finally apply to scattering

One preliminary:

Last time, noted that $G(r)$ not unique

Satisfies $(\nabla^2 + k^2) G = \delta^3(\vec{r})$

so does $G(r) + H(r)$ if $(\nabla^2 + k^2) H(r) = 0$

This should be reflected in integral equation

Really $$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0)\, \psi(\vec{r}_0)\, d^3r_0$$

ψ_0 = any solution to free particle eqn $(\nabla^2 + k^2)\psi_0 = 0$

Clearly solves $(\nabla^2 + k^2)\psi = \frac{2m}{\hbar^2} V(r)\, \psi$

Born approximation

For scattering, we're interest in large r
$|\vec{r}| \gg |\vec{r}_0|$ specifically

Note $|\vec{r}-\vec{r}_0|^2 = r^2 + r_0^2 - 2\vec{r}\cdot\vec{r}_0 \approx r^2\left(1 - 2\dfrac{\vec{r}\cdot\vec{r}_0}{r^2}\right)$

So $|\vec{r}-\vec{r}_0| \approx r\left(1 - \dfrac{\vec{r}\cdot\vec{r}_0}{r^2}\right) = r - \hat{r}\cdot\vec{r}_0$

Define $\vec{k} = k\hat{r}$ ($\sim$ local wave vector of scattered wave)

Then $e^{ik|\vec{r}-\vec{r}_0|} = e^{ikr} e^{-i\vec{k}\cdot\vec{r}_0}$

Outside exponentially, can be less accurate

$$\frac{1}{|\vec{r}-\vec{r}_0|} = \frac{1}{r}$$

Note, in exp, could have $k\hat{r}\cdot\vec{r}_0$ significant no matter how large r is

So we get

$$\psi \to \psi_0 - \frac{m}{2\pi\hbar^2} \underbrace{\frac{e^{ikr}}{r}}_{\text{scattered wave}} \int e^{-i\vec{k}\cdot\vec{r}_0} V(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3 r_0$$

We will take $\psi_0 = Ae^{ikz}$ = incident wave

To proceed, need to do something about ψ in integral.

If $V(r)$ is not very large, expect scattering to be weak

Then $\psi(r) \approx \psi_0(r)$

So plug in ψ_0 for ψ!

$$\psi \to Ae^{ikz} - \frac{m}{2\pi\hbar^2} A \frac{e^{ikr}}{r} \int e^{-i\vec{k}\cdot\vec{r}_0} e^{ikz} V(\vec{r}_0)\, d^3r_0$$

Define $\vec{k}' = k\hat{z}$

Then

$$\psi \to A\left[e^{ikz} + \left(-\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_0} V(\vec{r}_0)\, d^3r_0 \right) \frac{e^{ikr}}{r} \right]$$

Compare to standard scattering form

$$\psi \to A\left[e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \right]$$

See $\boxed{f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} V(\vec{r}_0)\, d^3r_0}$

= Born approximation
valid for weak V
Typically works well for high k
(when incident energy $\gg V$)

Complements partial wave method

Simplifies further if V confined to region $r < a$ and $ka \ll 1$

Then $e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} \approx 1$

$$f(\theta) \to -\frac{m}{2\pi\hbar^2} \underbrace{\int V(\vec{r}_0)\, d^3r_0}_{\approx \text{ "volume" of potential}}$$

For instance, soft sphere scattering

$$V(\vec{r}) = \begin{cases} V_0 & r<a \\ 0 & r>a \end{cases}$$

Then $f(\theta) \to -\frac{m}{2\pi\hbar^2} V_0 \left(\frac{4}{3}\pi a^3\right) = -\frac{2}{3}\frac{m a^3}{\hbar^2} V_0$

from this get differential & total cross section

$$D(\theta) = |f(\theta)|^2 \qquad \sigma = \int D(\theta)\, d\Omega$$

Actually, Born formula valid even if $V(\vec{r})$ not spherically symmetric

Generally $f(\theta, \phi)$

But if V is spherical, another simplification

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i\vec{\kappa}\cdot\vec{r}_0} V(\vec{r}_0)\, d^3 r_0$$

$\vec{\kappa} \equiv \vec{k}' - \vec{k}$

incident   scattered

$$= -\frac{m}{2\pi\hbar^2} \int e^{i\kappa r_0 \cos\theta_0} V(r)\, 2\pi \sin\theta_0\, d\theta_0\, r_0^2\, dr_0$$

$$= -\frac{m}{\hbar^2} \int r^2 V(r) \left[\int e^{i\kappa r_0 \cos\theta_0} \sin\theta_0\, d\theta_0\right] dr_0$$

did last time

$$= \frac{2\sin\kappa r_0}{\kappa r_0}$$

So $$\boxed{f(\theta) = -\frac{2m}{\hbar^2 \kappa} \int_0^\infty r_0 V(r_0) \sin\kappa r_0\, dr_0}$$

$$\kappa = |\vec{k}' - \vec{k}| = 2k\sin\frac{\theta}{2}$$