Lecture 31

Last time, developed Born approximation

General form: $f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{\varkappa}\cdot\vec{r}}\, V(\vec{r})\, d^3\vec{r}$

f: scattering amplitude

$\vec{\varkappa} = \vec{k}' - \vec{k}$

$\vec{k}' = k\hat{z}$ = incident wave vector

$\vec{k} = k\hat{r}$ = scattered wave vector

$\varkappa = 2k\sin\frac{\theta}{2}$

Start with another example

Yukawa scattering

$V(r) = \beta\frac{e^{-\mu r}}{r}$ = what Coulomb potential would be if photons had mass

= screened Coulomb potential (screening length $\frac{1}{\mu}$)

Since $V(r)$ spherically symmetric, use

$$f(\theta) = -\frac{2m}{\hbar^2\varkappa}\int_0^\infty r\,V(r)\sin\varkappa r\,dr$$

$$= -\frac{2m}{\hbar^2\varkappa}\beta\int_0^\infty e^{-\mu r}\sin\varkappa r\,dr$$

$$= -\frac{m}{\hbar^2\varkappa}\frac{\beta}{i}\int_0^\infty e^{(i\varkappa-\mu)r} - e^{(-i\varkappa-\mu)r}\,dr$$

$$= -\frac{m}{\hbar^2\varkappa}\frac{\beta}{i}\left[\frac{1}{i\varkappa-\mu}e^{(i\varkappa-\mu)r} + \frac{1}{i\varkappa+\mu}e^{-(i\varkappa+\mu)r}\right]\Bigg|_0^\infty$$

$$= +\frac{m}{\hbar^2\varkappa}\frac{\beta}{i}\left[\frac{1}{i\varkappa-\mu} + \frac{1}{i\varkappa+\mu}\right]$$

$$= \frac{m}{\hbar^2 \varkappa} \frac{\beta}{i} \left[\frac{(i\varkappa-\mu) + (i\varkappa+\mu)}{(i\varkappa-\mu)(i\varkappa+\mu)} \right]$$

$$= \frac{m}{\hbar^2 \varkappa} \frac{\beta}{i} \left[\frac{2i\varkappa}{-\varkappa^2-\mu^2} \right]$$

$$\boxed{f(\theta) = -\frac{2m}{\hbar^2} \frac{\beta}{\mu^2+\varkappa^2}} \qquad \text{(Prob 11.11)}$$

Might ask, is this valid?

$V(r) \to \infty$ at $r = 0$

Interpreting Born Approx.

See $f(\theta) \propto \int e^{i\vec{\varkappa}\cdot\vec{r}} V(\vec{r})\, d^3r$

$\sim$ Fourier transform of potential

Does this make sense?

Note $\hbar\vec{\varkappa}$ = momentum transferred by scattering

To understand, go back to perturbation theory

Think of $V(r)$ as perturbation H'

Recall expression for first order change in wave function:

$$\psi_n^{(1)} = \sum_{m\neq n} \frac{\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \psi_m^{(0)}$$

Recall $\psi_n^{(0)}$ is unperturbed wave function

Here interested in how $\psi_n^{(0)} = A e^{ikz}$ is affected

If $\psi_m^{(0)} = A e^{i \vec{k}_m \cdot \vec{r}}$, then each term in sum corresponds to wave scattered into momentum $\vec{p} = \hbar \vec{k}_m$

See that amplitude to scatter

$\vec{k} \to \vec{k}_m$ is proportional to $\langle e^{i\vec{k}_m \cdot \vec{r}} | V(r) | e^{i\vec{k}\cdot\vec{r}} \rangle$

$$\propto \int e^{i(\vec{k}-\vec{k}_m)\cdot\vec{r}} \, V(\vec{r}) \, d^3r$$

Similar to what we get in Born approx $\vec{k}_m \sim \vec{k}'$
(Can do carefully, get identical result)

General idea:
A potential $\propto e^{i\vec{k}'\cdot\vec{r}}$ can impart a momentum change $\hbar k'$ to a particle
(Basis of Bloch's theorem too)

So in Born approx, use Fourier transform to decompose $V(r) \to \int e^{i\vec{\varkappa}\cdot\vec{r}} \, V(\vec{\varkappa}) \, d^3\varkappa$

Each component $\vec{\varkappa}$ scatters particle into new momentum $\hbar(\vec{k} + \vec{\varkappa})$

This propagates away in direction Θ with

$$\varkappa = 2k \sin\frac{\Theta}{2}$$

This is a useful general principle:
Periodic potentials impart discrete momentum to particles
(same as diffraction gratings in optics)

Last point: Just like we can do higher order PT, can make higher order Born approx.

General statement: $\psi(\vec{r}) = \psi_0 + \int g(\vec{r}-\vec{r}_0)\, V(\vec{r}_0)\, \psi(\vec{r}_0)\, d^3 r_0$

$$g(\vec{r}) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}$$

0^{th} order: $\psi = \psi_0 = A e^{ikz}$

1^{st} order: Plug ψ_0 into integral

$$\psi_1 = \psi_0 + \int g(r-r_0)\, V(r_0)\, \psi_0(r_0)\, d^3 r_0$$

2^{nd} order: Plug ψ_1 into integral

$$\psi_2 = \psi_0 + \int g(r-r_0) V(r_0)\, \psi_0(r_0)\, d^3 r_0$$
$$+ \iint g(r-r_1) V(r_1)\, g(r-r_0) V(r_0)\, \psi_0(r_0)\, d^3 r_0\, d^3 r_1$$

and so forth

Sometimes useful... can do some tricky things by approximating each term in sum and then explicitly doing sum.