Lecture 33

Last time derived equation governing an interacting Bose condensate:

$$-\frac{\hbar^2}{2m}\nabla^2\phi + U(r)\phi + \frac{4\pi\hbar^2 a}{m} N|\phi|^2\phi = \mu\phi$$

a = scat. length

μ = chemical potential, to be determined

Called non-linear Schr. Eqn

also Gross-Pitaevski eqn.

Derivation puts together ideas from

- Identical particles
- Variational principle
- Scattering

You really are learning the tools to do real work

In general, need to solve equation numerically

But easy in one limit: $a>0$, large N

For $a>0$, interaction energy is positive
atoms repel each other

So for large N, expect size of condensate to expand.
Then typically potential energy also grows
assuming $U(r)$ increases with r

But $\nabla^2\phi$ term shrinks: ϕ varies more slowly than in non-interacting case

So in limit, $\nabla^2\phi$ term can be neglected

Leaves $\quad U\phi + \frac{4\pi\hbar^2 a}{m} N|\phi|^2\phi = \mu\phi$

Cancel ϕ, solve for $|\phi|^2$:

$$|\phi|^2 = \begin{cases} \frac{1}{N} \frac{m}{4\pi\hbar^2 a} \left[\mu - U(r)\right] & \text{for } U(r) < \mu \\ 0 & U(r) > \mu \end{cases}$$

Determine μ by enforcing $\int |\phi|^2 d^3r = 1$

Called Thomas-Fermi wavefunction
Agrees well with experiments.

Turn to last big thing we'll tackle:
Ch 9: Time dependent perturbation theory

Up until now, have focussed always on static problems.
Not so bad:
If we know eigenstates ψ_n & energies E_n,

then if $\psi(t=0) = \sum_n c_n \psi_n$, have

$$\psi(t) = \sum_n c_n \psi_n e^{-iE_n t/\hbar}$$

So a solution to static problem gives time dependence

But what if H itself is changing?
Example, spin-½ electron in oscillating B-field

Develop some tools to treat this case.

Mostly focus on two level system

Suppose unperturbed Hamiltonian H^0 with
two eigenstates ψ_a & ψ_b
energies E_a & E_b

So arbitrary state at time $t=0$ can be expressed

$$\psi(0) = c_a \psi_a + c_b \psi_b$$

with no perturbation, $\psi(t) = c_a \psi_a e^{-iE_a t/\hbar} + c_b \psi_b e^{-iE_b t/\hbar}$

But suppose we add a perturbation $H'(t)$

Still two levels, but now

$$\psi(t) = c_a(t)\, \psi_a e^{-iE_a t/\hbar} + c_b(t)\, \psi_b e^{-iE_b t/\hbar}$$

Problem is to determine c_a & c_b as funcs of time

Note, see already what can happen:
system can make transitions between states
Might have $c_a(0) = 1$, $c_b(0) = 0$
Later on, if $c_b(t) \approx 1$, say system has been driven to ψ_b

To proceed, use time-dep Schr. equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \qquad H = H^0 + H'(t)$$

Plugging in our ψ:

$$c_a E_a \psi_a e^{-iE_a t/\hbar} + c_a H' \psi_a e^{-iE_a t/\hbar} + c_b E_b \psi_b e^{-iE_b t/\hbar} + c_b H' \psi_b e^{-iE_b t/\hbar}$$

$$= i\hbar \left[\dot{c}_a \psi_a e^{-iE_a t/\hbar} - i\frac{E_a}{\hbar} c_a \psi_a e^{-iE_a t/\hbar} + \dot{c}_b E_b \psi_b e^{-iE_b t/\hbar} - i\frac{E_b}{\hbar} c_b \psi_b e^{-iE_b t/\hbar} \right]$$

Terms with E_a & E_b cancel:

$$c_a H' \psi_a e^{-iE_a t/\hbar} + c_b H' \psi_b e^{-iE_b t/\hbar} = i\hbar \left[\dot{c}_a \psi_a e^{-iE_a t/\hbar} + \dot{c}_b \psi_b e^{-iE_b t/\hbar} \right]$$

Isolate terms: take inner product with ψ_a:

$$c_a \underbrace{\langle \psi_a | H' | \psi_a \rangle}_{H'_{aa}} e^{-iE_a t/\hbar} + c_b \underbrace{\langle \psi_a | H' | \psi_b \rangle}_{H'_{ab}} e^{-iE_b t/\hbar} = i\hbar \dot{c}_a e^{-iE_a t/\hbar}$$

So $\quad \dot{c}_a = -\frac{i}{\hbar} \left[c_a H'_{aa} + c_b H'_{ab} e^{-i\omega_0 t} \right]$

$\omega_0 \equiv \frac{E_b - E_a}{\hbar}$ = Bohr frequency

Do same thing with ψ_b:

$$\dot{c}_b = -\frac{i}{\hbar} \left[c_b H'_{bb} + c_a H'_{ba} e^{i\omega_0 t} \right]$$

$H'_{bb} = \langle \psi_b | H' | \psi_b \rangle \qquad H'_{ba} = \langle \psi_b | H' | \psi_a \rangle = (H'_{ab})^*$

Very often, $H'_{aa} = H'_{bb} = 0$

Why? Usually ψ_a & ψ_b differ in some important quantum number

Examples: spin-$\frac{1}{2}$ electron: different m's
s-p states in H: different parity

To drive transition, apply operator that changes quantum number

electron: apply field along x, contains S_+ & S_-
atom: apply $V \sim \vec{r}$, changes parity

But then expectation value of operator in eigenstates must be zero:

electron, $\langle \uparrow | S_{\pm} | \uparrow \rangle = \langle \downarrow | S_{\pm} | \downarrow \rangle = 0$
atom, $\langle s | \vec{r} | s \rangle = \langle p | \vec{r} | \vec{p} \rangle = 0$

But occasionally need to keep diagonal terms.
Problem 9.4 shows how to handle.

For now, we'll drop them

$$\boxed{\begin{aligned} \dot{c}_a &= -\frac{i}{\hbar} c_b H'_{ab} e^{-i\omega_0 t} \\ \dot{c}_b &= -\frac{i}{\hbar} c_a H'_{ba} e^{i\omega_0 t} \end{aligned}}$$

Example: Problem 9.3
Suppose $H' = u\,\delta(t)$ $\quad u > 0$
and $c_a(t<0) = 1$, $\quad c_b(t<0) = 0$
Find $c_a(t)$, $c_b(t)$

A bit tricky:
Easiest to use $\delta(t) \to \begin{cases} \frac{1}{2\varepsilon} & -\varepsilon < t < \varepsilon \\ 0 & \text{else} \end{cases}$

and take $\varepsilon \to 0$ at end

So $\dot{c}_a$ & $\dot{c}_b = 0$ unless $-\varepsilon < t < \varepsilon$

For $t < -\varepsilon$, $\quad c_a = 1 \quad c_b = 0$
Say $t > \varepsilon$, $\quad c_a = a \quad c_b = b$

For $-\varepsilon < t < \varepsilon$:

$$\dot{c}_a = -\frac{i h}{2\varepsilon\hbar} e^{-i\omega_0 t} c_b \equiv -i\beta e^{-i\omega_0 t} c_b$$

$$\dot{c}_b = -\frac{i h}{2\varepsilon\hbar} e^{+i\omega_0 t} c_a \equiv -i\beta e^{+i\omega_0 t} c_a$$

$$\beta = \frac{h}{2\varepsilon\hbar}$$

Combine to one equation

$$\ddot{c}_b = -i\beta\left[i\omega_0 e^{i\omega_0 t} c_a + e^{i\omega_0 t} \dot{c}_a \right]$$

$$= -i\beta\left[i\omega_0 \left(\frac{1}{-i\beta} \dot{c}_b\right) + (-i\beta) c_b \right]$$

$$\ddot{c}_b = i\omega_0 \dot{c}_b - \beta^2 c_b$$

$$\ddot{c}_b - i\omega_0 \dot{c}_b + \beta^2 c_b = 0$$

Try $c_b = e^{\lambda t}$:

$$\lambda^2 - i\omega_0\lambda + \beta^2 = 0$$

$$\lambda = \frac{1}{2}\left(i\omega_0 \pm \sqrt{-\omega_0^2 - 4\beta^2}\right) = \frac{i}{2}(\omega_0 \pm \omega)$$

$$\omega = \sqrt{\omega_0^2 + 4\beta^2}$$

So $$c_b(t) = e^{i\frac{\omega_0 t}{2}}\left(A e^{i\frac{\omega t}{2}} - B e^{-i\frac{\omega t}{2}}\right)$$

$$c_b(-\varepsilon) = 0 \Rightarrow c_b(t) = A e^{i\frac{\omega_0 t}{2}}\left(e^{i\omega t/2} - e^{-i\omega\left(\frac{t}{2}+\varepsilon\right)}\right)$$

Plug in to get $c_a(t) = \frac{i}{\beta} e^{-i\omega_0 t} \dot{c}_b$

$$= -\frac{1}{2\beta} e^{-i\frac{\omega_0 t}{2}} A\left[(\omega+\omega_0) e^{i\omega t/2} + (\omega-\omega_0) e^{-i\omega(\varepsilon + t/2)}\right]$$

Impose $c_a(-\varepsilon) = 1$: Get $A = -\frac{\beta}{\omega} e^{i(\omega-\omega_0)\varepsilon/2}$

Then

$$C_a(t) = \frac{1}{2\omega} e^{-i\omega_0(t+\varepsilon)/2} \left[(\omega+\omega_0) e^{i\omega(t+\varepsilon)/2} + (\omega-\omega_0) e^{-i\omega(t+\varepsilon)/2} \right]$$

$$= e^{-i\omega_0(t+\varepsilon)/2} \left\{ \cos\left[\frac{\omega(t+\varepsilon)}{2}\right] + i \frac{\omega_0}{\omega} \sin\left[\frac{\omega(t+\varepsilon)}{2}\right] \right\}$$

and

$$C_b(t) = -2i \frac{\beta}{\omega} e^{i\omega_0(t-\varepsilon)/2} \sin\left[\frac{\omega(t+\varepsilon)}{2}\right]$$

Then

$$a = C_a(\varepsilon) = e^{-i\omega_0\varepsilon} \left[\cos(\omega\varepsilon) + i \frac{\omega_0}{\omega} \sin(\omega\varepsilon) \right]$$

$$b = C_b(\varepsilon) = -2i \frac{\beta}{\omega} \sin(\omega\varepsilon)$$

Finally let $\varepsilon \to 0$

$$\beta \to \infty \quad ; \quad \omega \to 2\beta = \frac{u}{\varepsilon\hbar}$$

So $a \to \cos\frac{u}{\hbar} + i \frac{\omega_0 \varepsilon\hbar}{u} \sin\frac{u}{\hbar} \to \boxed{\cos\frac{u}{\hbar}}$

$$\boxed{b \to -i\sin\frac{u}{\hbar}}$$