Lecture 34

Last time, started talking about time-dependent problem:

$$H = H^0 + H'(t)$$

Specify two-level system: ψ_a, ψ_b

$$\psi(t) = c_a(t)\,\psi_a\, e^{-iE_a t/\hbar} + c_b(t)\,\psi_b\, e^{-iE_b t/\hbar}$$

Apply time-dep Schr Eqn, get

$$\dot{c}_a = -\frac{i}{\hbar}\, c_b\, H'_{ab}\, e^{-i\omega_0 t}$$

$$\dot{c}_b = -\frac{i}{\hbar}\, c_a\, H'_{ba}\, e^{i\omega_0 t} \qquad \omega_0 = \frac{E_b - E_a}{\hbar}$$

We'll try to solve these

Started example: $H'_{ab} = u\,\delta(t)$

$c_a(0-) = 1$

$c_b(0-) = 0$

Wrote $H'_{ab} = \lim_{\varepsilon \to 0} \begin{cases} \frac{u}{2\varepsilon} & (-\varepsilon < t < \varepsilon) \\ 0 & (\text{else}) \end{cases}$

For finite ε, got

$$c_a(\varepsilon) = e^{-i\omega_0 \varepsilon}\left[\cos(\omega\varepsilon) + i\,\frac{\omega_0}{\omega}\sin\omega\varepsilon\right]$$

$$c_b(\varepsilon) = -2i\,\frac{\beta}{\omega}\sin\omega\varepsilon$$

$$\beta = \frac{u}{2\varepsilon\hbar}$$

$$\omega = \sqrt{\omega_0^2 + 4\beta^2}$$

To solve problem, take $\varepsilon \to 0$

So $\beta \to \infty$

$\omega \to 2\beta = \frac{u}{\varepsilon \hbar}$

$\omega\varepsilon \to \frac{u}{\hbar}$

So $c_a \to \cos \frac{u}{\hbar}$

$c_b \to -i \sin \frac{u}{\hbar}$

As function of u, system oscillates between ψ_a & ψ_b

This is pretty general result

But today mostly focus on approximate methods for small H'

Suppose again $c_a(0) = 1 \quad c_b(0) = 0$

To zero order in H', these don't change

To get first order, use these values on RHS of evolution eqns.

$$\dot{c}_a = -\frac{i}{\hbar} c_b(0) H'_{ab} e^{-i\omega_0 t} = 0$$

$$\dot{c}_b = -\frac{i}{\hbar} c_a(0) H'_{ba} e^{+i\omega_0 t} = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t}$$

So
$$\boxed{\begin{aligned} c_a^{(1)}(t) &= 1 \\ c_b^{(1)}(t) &= -\frac{i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt' \end{aligned}}$$

Should we worry about normalization?

It's OK: $|c_a^{(1)}|^2 + |c_b^{(1)}|^2 = 1 + O[(H')^2]$

only accurate to $O[H']$

Second order:

Use $c^{(1)}$'s on RHS of equations

$$\dot{c}_a^{(2)} = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t}\left[-\frac{i}{\hbar}\int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'\right]$$

$$\boxed{c_a^{(2)} = 1 - \frac{1}{\hbar^2}\int_0^t H'_{ab}(t') e^{-i\omega_0 t'} \int_0^{t'} H'_{ab}(t'') e^{i\omega_0 t''} dt''\, dt'}$$

and $c_b^{(2)} = c_b^{(1)}$, since $c_a^{(1)} = c_a^{(0)}$

Check our δ-fcn example: $H'_{ab} = u\,\delta(t-\varepsilon)$

(make pulse after $t=0$)

$$c_b^{(2)} = -\frac{i}{\hbar}\int_0^t u\,\delta(t'-\varepsilon) e^{i\omega_0 t'} dt'$$

$$= -i\frac{u}{\hbar}, \quad \text{first order expansion of } -i\sin\frac{u}{\hbar}$$

$$c_a^{(2)} = 1 - \frac{u^2}{\hbar^2}\int_0^t \underbrace{\delta(t'-\varepsilon) e^{-i\omega_0 t'}}_{\text{integral gives 1, and } t'=\varepsilon} \int_0^{t'} \delta(t''-\varepsilon) e^{i\omega_0 t''} dt''\, dt'$$

$$= 1 - \frac{u^2}{\hbar^2}\int_0^{\varepsilon} \delta(t''-\varepsilon) e^{i\omega_0 t''} dt''$$

Note $\int_0^a \delta(x-a)\,dx = \frac{1}{2}$

$$c_a^{(2)} = 1 - \frac{1}{2}\frac{u^2}{\hbar^2}, \quad 2^{nd} \text{ order expansion of } \cos\frac{u}{\hbar}$$

Makes sense.

Very often have sinusoidal perturbation

$$H'_{ab} = V_{ab} \cos \omega t$$

To first order,

$$c_b^{(1)} = -\frac{i}{\hbar} V_{ba} \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt'$$

$$= -\frac{i}{2\hbar} V_{ba} \int_0^t e^{i(\omega+\omega_0)t'} + e^{i(\omega_0-\omega)t'} dt'$$

$$= -\frac{i}{2\hbar} V_{ba} \left[\frac{e^{i(\omega+\omega_0)t'}}{i(\omega+\omega_0)} + \frac{e^{i(\omega_0-\omega)t'}}{i(\omega_0-\omega)} \right]\Big|_0^t$$

$$= -\frac{V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \underbrace{\frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}} \right]$$

See that $c_b^{(1)}$ is largest if $\omega_0 \approx \omega$
- driving on resonance

Define $\omega - \omega_0 = \Delta$, assume $|\Delta| \ll \omega_0$

Then first term is small, neglect.

Second term : $\frac{e^{-i\Delta t}-1}{-\Delta} = \frac{1}{\Delta} e^{-i\Delta t/2} \left(e^{i\Delta t/2} - e^{-i\Delta t/2} \right)$

$$= \frac{2i}{\Delta} e^{-i\Delta t/2} \sin \frac{\Delta t}{2}$$

So $c_b^{(1)} = -i \frac{V_{ba}}{\hbar \Delta} e^{-i\Delta t/2} \sin \frac{\Delta t}{2}$

Probability to make transition from $a \to b$ is

$$\boxed{P_{a\to b} = |c_b^{(1)}|^2 = \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2 \Delta t/2}{\Delta^2}}$$