Lecture 35

Last time, developed time-dependent PT

$H = H^0 + H'(t)$, H' small

Consider two level system ψ_a, ψ_b, with $\langle a|H'|a\rangle = \langle b|H'|b\rangle = 0$

$$\psi(t) = c_a(t)\,\psi_a + c_b(t)\,\psi_b\, e^{-i\omega_0 t} \qquad \omega_0 = \frac{E_b - E_a}{\hbar}$$

$$c_a(0) = 1 \qquad c_b(0) = 0$$

Then

$$c_a(t) \approx 1 - \frac{1}{\hbar^2}\int_0^t H'_{ab}(t')\, e^{-i\omega_0 t'}\left[\int_0^{t'} H'_{ba}(t'')\, e^{i\omega_0 t''}\, dt''\right] dt'$$

$$c_b(t) \approx -\frac{i}{\hbar}\int_0^t H'_{ba}(t')\, e^{i\omega_0 t'}\, dt'$$

In particular, if $H'_{ab} = V_{ab}\cos\omega t$, with $\omega \approx \omega_0$

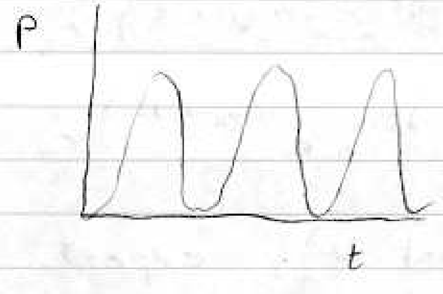
get

$$c_b(t) \approx -i\,\frac{V_{ba}}{\hbar}\, e^{-i\Delta t/2}\, \frac{\sin\frac{\Delta t}{2}}{\Delta} \qquad \Delta = \omega - \omega_0$$

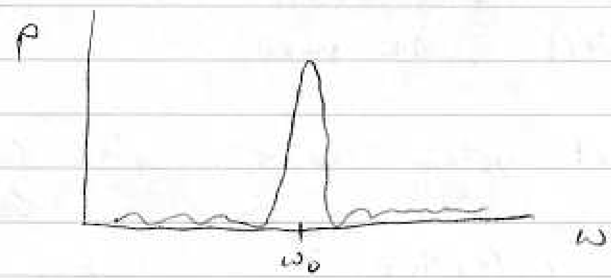
Transition probability

$$P_{a\to b}(t) = |c_b|^2 \approx \frac{|V_{ba}|^2}{\hbar^2}\,\frac{\sin^2\frac{\Delta t}{2}}{\Delta^2}$$

As function of t, this oscillates:



As function of ω, peaked at ω_0



At $\omega = \omega_0$, $\quad P = \frac{|V_{ab}|^2}{\hbar^2} \frac{t^2}{4}$ grows without limit as $t \to \infty$

But perturbation theory is only valid so long as $P \ll 1$: breaks down for large t.

For a true two-level system, can actually do better: solve exactly (in limit small V_0 and $\omega \approx \omega_0$)

Problem 9.7. Important result, I'll do here.

Start with evolution equations:

$$\dot{c}_a = -\frac{i}{\hbar} V_{ab} \cos\omega t \, e^{-i\omega_0 t} c_b$$

$$\dot{c}_b = -\frac{i}{\hbar} V_{ba} \cos\omega t \, e^{i\omega_0 t} c_a$$

Rewrite, define $\Omega = \frac{V_{ab}}{\hbar}$

$$\dot{c}_a = -i \frac{\Omega}{2} \left(e^{-i(\omega_0 - \omega)t} + e^{-i(\omega_0 + \omega)t} \right) c_b$$

$$\dot{c}_b = -i \frac{\Omega^*}{2} \left(e^{i(\omega_0 + \omega)t} + e^{i(\omega_0 - \omega)t} \right) c_a$$

Now if $\Omega \ll \omega_0$ (weak V), expect c_a & c_b to vary slowly since $\dot{c}_a$ & $\dot{c}_b$ are small

But if $\omega \approx \omega_0$, then $e^{i(\omega_0+\omega)t} \approx e^{2i\omega_0 t}$
oscillates quickly

Gives component of $\dot{c}$ that rapidly alternates sign
Tends to average out, not much effect.

While $e^{i(\omega_0-\omega)t}$ component is slowly varying
adds up over time, get big effect

Approximation:
Drop rapidly oscillating terms
Called rotating wave approx.

Then

$$\dot{c}_a \approx -i\frac{\Omega}{2}e^{+i\Delta t}c_b$$

$$\dot{c}_b \approx -i\frac{\Omega^*}{2}e^{-i\Delta t}c_a$$

Combine as before

$$\ddot{c}_b = -i\frac{\Omega^*}{2}e^{-i\Delta t}\left(-i\Delta c_a + \dot{c}_a\right)$$

$$= -i\frac{\Omega^*}{2}e^{-i\Delta t}\left[-i\Delta\cdot\left(-\frac{2}{i}\frac{1}{\Omega^*}e^{+i\Delta t}\dot{c}_b\right) + \left(-i\frac{\Omega}{2}e^{+i\Delta t}c_b\right)\right]$$

$$= -i\Delta\dot{c}_b - \frac{|\Omega|^2}{4}c_b$$

$$\boxed{\ddot{c}_b + i\Delta\dot{c}_b + \frac{|\Omega|^2}{4}c_b = 0}$$

Solve as before, try $c_b = e^{\lambda t}$

$$\lambda^2 + i\Delta\lambda + \frac{|\Omega|^2}{4} = 0$$

$$\lambda = \frac{1}{2}\left(-i\Delta \pm \sqrt{-\Delta^2-|\Omega|^2}\right) = -\frac{i}{2}\left(\Delta \pm \sqrt{\Delta^2+|\Omega|^2}\right)$$

Define $\frac{1}{2}\sqrt{\Delta^2+|\Omega|^2} = \omega_r$ Rabi frequency

(ok, I know Rabi)

So $c_b(t) = e^{-i\Delta t/2}\left(Ae^{i\omega_r t} + Be^{-i\omega_r t}\right)$

$c_b(0) = 0$ so $A = -B$

$$c_b = Ae^{-i\Delta t/2}\left(e^{i\omega_r t} - e^{-i\omega_r t}\right)$$

$$= 2iA\, e^{-i\Delta t/2} \sin\omega_r t$$

Also $c_a(0) = \frac{2i}{\Omega^*} e^{i\Delta t}\, \dot{c}_b$

$$\frac{2i}{\Omega^*} A \left[i\left(-\frac{\Delta}{2}+\omega_r\right) - i\left(-\frac{\Delta}{2}-\omega_r\right)\right] = 1$$

$$\left[i \cdot 2\omega_r \right]$$

$$-\frac{4}{\Omega^*} A \,\omega_r = 1$$

$$A = -\frac{\Omega^*}{4\omega_r}$$

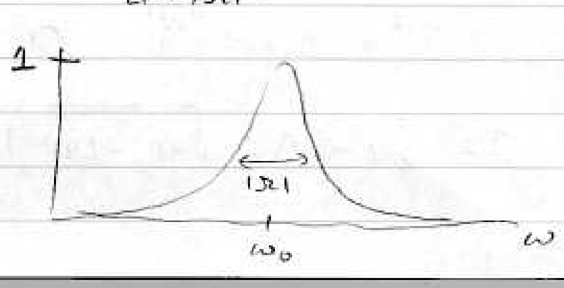
$$\boxed{C_b(t) = -i\frac{\Omega^*}{2\omega_r} e^{-i\Delta t/2} \sin\omega_r t}$$

$$P_{a\to b} = \frac{|\Omega|^2}{4\omega_r^2}\sin^2\frac{\omega_r t}{2} = \frac{|\Omega|^2}{\Delta^2+|\Omega|^2}\sin^2\omega_r t$$

As function of time, oscillates at ω_r

Max transition prob = $\frac{|\Omega|^2}{\Delta^2+|\Omega|^2}$

Lorentzian:



Compare to PT result

PT valid if $P_{max} \ll 1$

$\Rightarrow \quad |\Omega|^2 \ll \Delta^2$

Then $\quad P \approx \dfrac{|\Omega|^2}{\Delta^2} \sin^2 \omega_r t$

$\omega_r \approx \frac{1}{2} \Delta$

$= \dfrac{|V_{ab}|^2}{\hbar^2 \Delta^2} \sin^2 \dfrac{\Delta t}{2}$, same as PT.