Lecture 36

Continuing with time dependent PT

For sinusoidal driving field, have

\[ P_{\text{abs}}(t) = \frac{1}{\pi} \frac{\sin^2 \Delta t}{\Delta t^2} \quad \Delta = \omega_1 - \omega_2 \]

for any system, as long as \( P_{\text{max}} \ll 1 \)

or \( P_{\text{abs}}(t) = \frac{1}{\pi} g^2 \sin^2 \omega t \) for two-level system, \( \omega_1, \omega_2 \ll \omega_0 \)

and

\[ \omega_0 = \frac{1}{\Delta} \left( \frac{1}{\pi} \frac{1}{\Delta} \right) \]

Today, consider specifically driving transitions in atoms with an E+M field.

\[ H^0 = \text{atomic Hamiltonian (Hydrogen)} \]

\[ H' = \text{energy of electron in E+M field} \]

Neglecting B-field for now, \( E' = \text{energy in E-field} \)

\[ H' = -e \phi \quad \phi = \text{electric potential} \]

\[ \phi = -\frac{\mathcal{E} \cdot \mathbf{d}^2}{\mathcal{E}} \]

Say \( \mathbf{E} = E_0 \hat{z} \cos(ky - \omega t) \): plane wave polarized along \( z \), travelling along \( y \)

Then \( \phi = -E_0 \hat{z} \cos(ky - \omega t) \)

\[ H' = eE_0 \hat{z} \cos(ky - \omega t) \]
We need $H_{6e} = e E_0 \leq 24 \frac{1}{2} \cos(k y - \omega t) |24_a)\$

Now, electron in atom is localized to $w = 5 \times 10^{-6} \text{m}$

$\cos k y$ varies over length $\frac{1}{k} = \lambda$

For visible light, $\lambda \approx 5 \times 10^{-7} \text{m} \gg a$

So spatial variation of cosine is not significant

Approximate $y \to y_0 = \text{location of atom}$
for simplicity, take $y_0 = 0$

Then $H_{6e} = e E_0 \cos \omega t \left( \two 1 2 \middle| 24_c \right)\$

Define $y_0 = \left( \two 1 2 \middle| 24_e \right) = \text{matrix element of dipole moment}$

"dipole approximation"

$\hat{\rho} = y_0^2$

$H_{6e} = -\hat{\rho} E_0 \cos \omega t$  like before, $V_{6e} = -\hat{\rho} E_0$

So we get

$P_{\text{max}} = \left| \frac{2 E_0}{h} \right|^2 \frac{\sin^2 20^\circ}{\Delta^2}$ in perturbative limit.

That is very handy if you are driving a transistor using a laser.

However, very often you don't have a laser, you have light from a lamp (or the sun, or a floor, etc.)

Then we don't have one well defined $\omega$, we have a whole range.
Cell such illumination incoherent

Hard to write down $E$, instead use energy density:

For monochromatic light, $n = \frac{E_0}{c} \frac{E}{2}$ and $\frac{\nu}{c} = \frac{1}{\lambda}$

See $P_{\text{avg}} = \frac{2n}{\varepsilon_0 \lambda^2} \int_0^L \frac{\sin^2 \frac{\Delta \lambda}{2}}{\Delta \lambda^2} d\lambda$

For incoherent light:

$$n = \int p(\omega) \, d\omega$$

$p(\omega)$ = spectral energy density

$p(\omega) \, d\omega \, d^3r$ = energy in volume $d^3r$ with freq between $\omega$ and $\omega + d\omega$

Each increment $d\omega$ contributes to $P$, so

$$P_{\text{avg}} = \frac{2}{\varepsilon_0 \lambda^2} \int_0^L \frac{\sin^2 \frac{\Delta \lambda}{2}}{\Delta \lambda^2} p(\omega) \, d\omega$$

Now $\frac{\sin^2 \frac{\Delta \lambda}{2}}{\Delta \lambda^2}$ is sharply peaked at $\omega = \omega_0$ if $\Delta \omega \ll 1$

So only values of $\rho$ near $\omega_0$ will be important, approximate $\rho$ as constant:

$$P = \frac{2}{\varepsilon_0 \lambda^2} \int_0^L \rho(\omega_0) \frac{\sin^2 \frac{\Delta \lambda}{2}}{\Delta \lambda^2} \, d\lambda$$
Change variables \( v = \frac{\Delta t}{2} \) \( dv = \frac{1}{2} dt \)

Then

\[
\rho = \frac{1}{\varepsilon_0 c^2} \left| p \right|^2 \rho(\omega_0) \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty \frac{\sin^2 v}{v^2} \, dv
\]

Look up definite integral = \( \pi \)

So

\[
\rho = \frac{\pi}{\varepsilon_0 c^2} \left| p \right|^2 \rho(\omega_0) t
\]

Grows linearly in time

Note, in monochromatic form, with \( \Delta = 0 \), \( \rho \) grows quadratically in time, these lines, because for longer times a narrower range of \( \omega \) contributes.

Typically define transition rate

\[
R = \frac{d\rho}{dt} = \frac{\pi}{\varepsilon_0 c^2} \left| p \right|^2 \rho(\omega_0)
\]

Polarization effects:
We've assumed light is polarized along \( \hat{z} \), natural light typically unpolarized.

Really

\[
R_\hat{z} = \frac{\pi}{\varepsilon_0 c^2} \left| p_\hat{z} \right|^2 \rho(\omega_0)
\]

= transition rate for \( \hat{z} \), polarized light

\( p_\hat{z} \) = \( z \) comp of dipole moment

\( \rho_\hat{z} \) = density of \( \hat{z} \) polarized light
Get some form for your polarizations.

By symmetry, \( \| \mathbf{\rho} \|^2 = \| \mathbf{\rho}_1 \|^2 = \| \mathbf{\rho}_2 \|^2 \) (atoms have no preferred direction)

... (content continues)

Then
\[
\mathbf{R}_i = \frac{\pi}{3 \varepsilon_0 c^2} \| \mathbf{\rho} \|^2_{\| \mathbf{\rho}_{i \omega} \|} \mathbf{p}_{i \omega} \]  
(\( i \) th component)

Total rate
\[
\mathbf{R} = \sum \mathbf{R}_i = \frac{\pi}{3 \varepsilon_0 c^2} \| \mathbf{\rho} \|^2 \sum \mathbf{p}_{i \omega} \]

\[\mathbf{R} = \frac{\pi}{3 \varepsilon_0 c^2} \| \mathbf{\rho} \|^2 \mathbf{p}_{\omega} \]

For \( \mathbf{\rho} \) = total energy density

... (content continues)

For an example of calculating \( \mathbf{\rho} \), consider Problem 9.1

Calculate \( \langle \mathbf{J}_{1z} \mathbf{J}_{2z} \rangle \) between ground state \( n=1 \) and excited states \( n=2 \)

Ignore spin:

\( n \) doesn’t couple to spin—directly
Spin-orbit coupling would actually change what excited states are, but averages out to some overall effect.
So ground state is 1s, excited states 2s, 2p with $m = 0, \pm 1$

$\langle 200 | z | 100 \rangle = 0$, since $2s, 2p$ are even in $z$

$\langle 211 | z | 110 \rangle = \int R_2(r) \psi_1^*(r) \psi_0(r) \, d^3r$

But $z = r \cos \theta$, no $\phi$ dependence

$\psi_0 = \frac{i}{\sqrt{\pi}}$

$\psi_1^* = e^{i \phi}$

So integral $= \int e^{i \phi} \, d\phi = 0$

Similarly $\langle 211 | z | 100 \rangle = 0$

Then

$\langle 210 | z | 110 \rangle = \int R_{1\ell}(r) \psi_0^* \psi_1 \, d^3r$

$= R_{1\ell}(r) \frac{2}{a \pi} e^{-r/\alpha}$

$z = r \cos \theta$

$R_{1\ell}(r) = \frac{1}{\sqrt{24 \pi a^2 \alpha}} \frac{1}{a^\ell} \frac{1}{\alpha} e^{-r/\alpha}$

$\psi_0 = \frac{1}{\sqrt{4 \pi}}$

$\psi_1 = \frac{1}{\sqrt{4 \pi}} \cos \theta$

Nice to break into radial part and angular part

Angular part:

$\int \frac{\ell!}{4 \pi} \cos^\ell \theta \sin \theta \, d\theta \, d\phi$

$= \frac{\ell!}{4 \pi} \frac{\cos^\ell \theta}{\ell} |_0^\pi$

$= \frac{\ell!}{2^\ell} \frac{1}{\ell}$

$= \frac{\ell!}{2^\ell} \frac{\pi}{2} \frac{2}{\ell}$

$= \frac{1}{\sqrt{5}}$
Radial part

\[ \frac{1}{4\pi} \frac{1}{a^2} \int_0^a \left( e^{-\frac{r}{a}} \right)^2 (r^2) r^2 dr \]

\[ = \frac{1}{4\pi} \frac{1}{a^4} \int_0^a r^4 e^{-\frac{r}{a}} \, dr \]

\[ = \frac{1}{4\pi} \frac{1}{a^4} 4! \left( \frac{2a}{3} \right)^5 \]

\[ = 4 \cdot \frac{\sqrt{6}}{a} \left( \frac{2}{3} \right)^5 \]

\[ s_0 \langle 210 | 21100 \rangle = 4 \cdot \frac{\sqrt{6}}{a} \left( \frac{2}{3} \right)^5 \]

\[ = \frac{2^{15/2}}{3^5} a = 0.745a \]