Lecture 37

Last time, considered E&M transitions in atoms

Monochromatic light amplitude E_0 : $P_{abs} = \left| \frac{\vec{\wp}\cdot\vec{E}_0}{\hbar} \right|^2 \frac{\sin^2 \frac{\Delta t}{2}}{\Delta^2}$

Incoherent light, spectral density ρ

$$R_{abs} = \frac{\pi}{3\varepsilon_0 \hbar^2} |\vec{\wp}|^2 \rho(\omega_0)$$

$\vec{\wp} = -e\vec{r}$ = dipole moment operator

Today, tell rest of the story

In particular, discuss spontaneous emission

Atom in excited state, say 2p state of H
can spontaneously decay to a lower energy state

This should be surprising...
2p state is an eigenstate

If we don't apply a perturbation, state
shouldn't change

Why does it?

Need to develop quantum theory of E&M field to
answer.

That takes some work, & Griffith's doesn't cover it

So I'll just give the idea

Say atom was in a box with perfectly conducting walls.

Then only certain modes of EM field allowed

$$\vec{E}(\vec{r},t) = A\hat{\varepsilon} \sin\frac{n_x \pi x}{L_x} \sin\frac{n_y \pi y}{L_y} \sin\frac{n_z \pi z}{L_z} \cos\omega t$$

$$\vec{k} = \pi\left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z}\right) \qquad \omega = \frac{|\vec{k}|}{c}$$

Just like particle in a box!

Consider a single mode (n_x, n_y, n_z)

Energy of mode $= \int u\, d^3r \qquad u(r) = \frac{1}{2\varepsilon_0} |\vec{E}(\vec{r})|^2$

$$= \frac{|A|^2}{2\varepsilon_0} \frac{V}{8}$$

In quantum E&M, amplitude A itself becomes quantum variable

Hamiltonian $\quad H = \frac{V}{16\varepsilon_0} |A|^2$

Actually need to let A be complex

if $A = |A| e^{i\phi}$, $\quad E \propto |A| \cos(\omega t + \phi)$
$= \mathrm{Re}[A e^{i\omega t}]$

Use same trick in classical optics, if you've seen it

Say $A = q + ip$, q & p both → quantum operators

Then $\quad H = \frac{V}{16\varepsilon_0}(q^2 + p^2)$

Just like Hamiltonian for harmonic oscillator

$\Rightarrow$ Each mode of field acts just like quantum harmonic oscillator

Allowed energies $E = \hbar\omega(n+\frac{1}{2})$
$\Rightarrow$ photons

Get all kind of interesting things, but what I wanted to point out is zero point energy: $\frac{\hbar\omega}{2}$

Corresponds to non-zero zero-point field E_0

So even in empty box w/ no photons, still have quantum electric field noise.

It turns out, this noise can drive atomic transitions.

Only from excited state $\rightarrow$ lower state
(atom can emit energy into field, can't absorb any)

Called spontaneous emission

Can see how it works

Have $R = \frac{\pi}{3\varepsilon_0\hbar^2}|\wp|^2\rho(\omega_0)$

Here $\rho(\omega_0) = \frac{\frac{1}{2}\hbar\omega_0}{V} \times$ (# of states between ω_0 & $\omega_0+d\omega$)

Work out just as for electrons in a box

$= \frac{V}{\pi^2 c^3}\omega_0^2$

$$R_{sp} = \frac{\pi}{3\varepsilon_0\hbar^2} |\vec{p}|^2 \cdot \frac{\hbar\omega_0}{2V} \cdot \frac{V}{\pi^2 c^3} \omega_0^2$$

$$= \frac{1}{6\pi\varepsilon_0} |\vec{p}|^2 \frac{\omega_0^3}{\hbar c^3}$$

Turns out, real answer is twice this

$$R_{sp} = \frac{1}{3\pi\varepsilon_0\hbar} |\vec{p}|^2 \left(\frac{\omega_0}{c}\right)^3$$

Hand-waving can only get you so far in life...
To do correctly, either develop quantum E&M for real,
or can get from statistical argument (Section 9.3)

So generally, two ways to drive a transition

Stimulated: apply a field, it drives (reversible) $\psi_a \leftrightarrow \psi_b$

Spontaneous: "vacuum" field drives $\psi_b \rightarrow \psi_a$
(irreversible)

Problem first considered by Einstein
sometimes still see his notation

write $R_{sp} = A \equiv$ Einstein A coefficient

$R_{stim} = B\rho(\omega_0)$, Einstein B coefficient

$$\text{So } A = \frac{|\vec{p}|^2\omega_0^3}{3\pi\varepsilon_0\hbar c^3}$$

$$B = \frac{\pi|\vec{p}|^2}{3\varepsilon_0\hbar^2}$$

So lets say we have a collection of atoms

ψ_a = ground state

ψ_b = excited state

We apply radiation with given $\rho(\omega_0)$

Can solve for state populations very easily

Say N_a atoms in ψ_a

N_b in ψ_b $\qquad N = N_a + N_b$ total

$$\frac{dN_b}{dt} = \underbrace{-AN_b}_{\text{spontaneous emission}} \underbrace{- BN_b\,\rho(\omega_0)}_{\text{stim emission}} + \underbrace{BN_a\,\rho(\omega_0)}_{\text{stim absorbtion}}$$

In equilibrium, $\frac{dN_b}{dt} = 0$

also $N_a = N - N_b$

Then $-AN_b - BN_b\rho + BN\rho - BN_b\rho = 0$

$$N_b(A + 2B\rho) = NB\rho$$

$$\boxed{\frac{N_b}{N} = \frac{B\rho}{A + 2B\rho}}$$

Gives population in excited state as function of ρ

Note, this only works for incoherent radiation

monochromatic radiation gives Rabi oscillations,

more complicated with sponteneous emission

Finally, spontaneous emission sets lifetime of excited state

If $\rho = 0$, no applied field,

$$\frac{dN_3}{dt} = -AN_3 \quad \Rightarrow \quad N_3(t) = N_3(0)\, e^{-At}$$

Exponential decay; "lifetime" $\tau = \frac{1}{A} = \frac{3\pi \varepsilon_0 \hbar c^3}{|\vec{p}|^2 \omega_0^3}$

For instance, lifetime of ψ_{210} state of hydrogen

Here $p_z = -e\langle \psi_{210} | z | \psi_{100} \rangle$

$$= -ea\left(\frac{2^{15/2}}{3^5}\right) \quad \text{from last time}$$

For this state, $p_x = p_y = 0$ (ψ_{210} & ψ_{100} indep of ϕ)

$$|\vec{p}|^2 = p_z^2 = e^2 a^2 \frac{2^{15}}{3^{10}}$$

$$\tau = \frac{3\pi \varepsilon_0 \hbar c^3}{\omega_0^3} \cdot \frac{3^{10}}{2^{15}} \frac{1}{e^2 a^2}$$

$$\omega_0 = \frac{1}{\hbar}(E_2 - E_1) = \frac{1}{\hbar}(-E_1)\left(1 - \frac{1}{4}\right)$$

$$= -\frac{E_1}{\hbar}\frac{3}{4}$$

$$\tau = \frac{3^{11}}{2^{15}} \pi \left(\frac{4}{3}\right)^3 \frac{\varepsilon_0 \hbar^4 c^3}{(-E_1)^3} \frac{1}{e^2 a^2}$$

Use $\frac{e^2}{4\pi\varepsilon_0} = \hbar c \alpha$

$a = \frac{\hbar}{mc\alpha}$ Get $\tau = \frac{3^8}{2^9} \frac{1}{\alpha^3} \frac{\hbar}{(-E_1)}$

$E_1 = -\frac{\alpha^2 mc^2}{2}$ $\frac{\hbar}{-E_1} = 2 \cdot 10^{16}\ s^{-1}$

$\alpha = \frac{1}{137}$

So $\boxed{\tau = 1.6 \times 10^{-9}\ s}$

Get same answer for other 2p states

Note, this means energy of 2p state is intrinsically uncertain:

$$\Delta E \approx \frac{\hbar}{\tau} = 4 \times 10^{-7}\ eV$$

$$\frac{\Delta E}{h} = 100\ MHz$$