Supplement 4 - Partial Wave Expansion of e^{ikz}

In spherical coordinates, $e^{ikz} = e^{ikr\cos\theta}$

Let $x = kr$

Know we can write $e^{ix\cos\theta} = \sum_\ell c_\ell j_\ell(x) P_\ell(\cos\theta)$

Since j_ℓ is only solution to radial Sch. Eqn. that is regular at origin

Use orthogonality of P_ℓ's: $\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta = \frac{2\delta_{\ell\ell'}}{2\ell+1}$

$$\int P_\ell(\cos\theta) e^{ix\cos\theta} \sin\theta \, d\theta = \frac{2}{2\ell+1} c_\ell j_\ell(x)$$

Now we could just set $x=0$, and get

$$c_\ell = \frac{2\ell+1}{2} \frac{1}{j_\ell(0)} \int P_\ell(\cos\theta) \sin\theta \, d\theta$$

except that integral and $j_\ell(0)$ are both zero for $\ell > 0$

Instead, note $j_\ell(x) \to \frac{2^\ell \ell!}{(2\ell+1)!} x^\ell$ for small x (Table 4.4)

So $\left(\frac{d}{dx}\right)^\ell j_\ell(x) = \frac{2^\ell (\ell!)^2}{(2\ell+1)!}$

also have $\left(\frac{d}{dx}\right)^\ell e^{ix\cos\theta} = i^\ell \cos^\ell\theta \, e^{ix\cos\theta}$

So, differentiating both sides and then setting $x=0$,

$$\int P_\ell(\cos\theta) \, i^\ell \cos^\ell\theta \sin\theta \, d\theta = \frac{2^{\ell+1}(\ell!)^2}{(2\ell)!} \frac{1}{(2\ell+1)^2} c_\ell$$

$$c_\ell = \frac{(2\ell)!}{2^{\ell+1}(\ell!)^2} (2\ell+1)^2 i^\ell \int P_\ell(\cos\theta) \cos^\ell\theta \sin\theta \, d\theta$$

Define $y = \cos\theta$

Integral becomes $I = \int_{-1}^{1} y^{\ell} P_{\ell}(y)\, dy$

Use Rodrigues formula, Eq 4.28

$$P_{\ell}(y) = \frac{1}{2^{\ell}\ell!}\left(\frac{d}{dy}\right)^{\ell}(y^2-1)^{\ell}$$

$$I = \frac{1}{2^{\ell}\ell!}\int_{-1}^{1} y^{\ell}\left(\frac{d}{dy}\right)^{\ell}(y^2-1)^{\ell}\, dy$$

Integrate by parts: $u = y^{\ell}$ $\quad dv = \left(\frac{d}{dy}\right)^{\ell}(y^2-1)^{\ell}$

$du = \ell y^{\ell-1}$ $\quad v = \left(\frac{d}{dy}\right)^{\ell-1}(y^2-1)^{\ell}$

$$I = \frac{1}{2^{\ell}\ell!}\left[y^{\ell}\underbrace{\left(\frac{d}{dy}\right)^{\ell-1}(y^2-1)^{\ell}}\Big|_{-1}^{1} - \ell\int_{-1}^{1} y^{\ell-1}\left(\frac{d}{dy}\right)^{\ell-1}(y^2-1)^{\ell}\, dy\right]$$

Since power of y^2-1 is greater than power of $\frac{d}{dy}$, will have y^2-1 term at end $\to 0$

$$I = \frac{1}{2^{\ell}\ell!}(-\ell)\int y^{\ell-1}\left(\frac{d}{dy}\right)^{\ell-1}(y^2-1)^{\ell}\, dy$$

Repeating ℓ times, see that

$$I = \frac{1}{2^{\ell}\ell!}(-1)^{\ell}\,\ell!\int_{-1}^{1}(y^2-1)^{\ell}\, dy$$

$$= \frac{1}{2^{\ell}}\int_{-1}^{1}(1-y^2)^{\ell}\, dy$$

To do this one, use integration by parts again

In general, consider $J_{mn} = \int_{-1}^{1} y^m (1-y^2)^n\, dy$

$u = (1-y^2)^n$ $\quad dv = y^m$

$du = -2yn(1-y^2)^{n-1}$ $\quad v = \frac{y^{m+1}}{m+1}$

So $J_{mn} = \underbrace{(1-y^2)^n \frac{y^{m+1}}{m+1} \Big|_{-1}^{1}}_{=0} + 2\frac{n}{m+1} \int_{-1}^{1} y^{m+2} (1-y^2)^{n-1} \, dy$

$= 2 \frac{n}{m+1} J_{(m+2)(n-1)}$

We have $I = \frac{1}{2^\ell} J_{0\ell}$

$= \frac{1}{2^\ell} \, 2 \, \frac{\ell}{1} \, J_{(2)(\ell-1)}$

$= \frac{1}{2^\ell} \, 2^2 \, \frac{\ell(\ell-1)}{1 \cdot 3} \, J_{(4)(\ell-2)}$

repeating, $= \frac{1}{2^\ell} \, 2^\ell \, \frac{\ell!}{1 \cdot 3 \cdot 5 \cdots (2\ell-1)} \, J_{(2\ell)(0)}$

Note $1 \cdot 3 \cdot 5 \cdots (2\ell-1) = \frac{(2\ell)!}{2 \cdot 4 \cdot 6 \cdots 2\ell} = \frac{(2\ell)!}{2^\ell \ell!}$

So $I = \frac{1}{2^\ell} \, 2^\ell \, \frac{2^\ell (\ell!)^2}{(2\ell)!} \int_{-1}^{1} y^{2\ell} \, dy$

$= \frac{2^\ell (\ell!)^2}{(2\ell)!} \, \frac{2}{2\ell+1}$

Plugging back in, get

$c_\ell = \frac{(2\ell)!}{2^{\ell+1} (\ell!)^2} (2\ell+1)^2 \, i^\ell \cdot \frac{2^{\ell+1} (\ell!)^2}{(2\ell)!} \, \frac{1}{2\ell+1}$

$= (2\ell+1) \, i^\ell$

Thus $e^{ikz} = \sum_\ell i^\ell (2\ell+1) \, j_\ell(kr) \, P_\ell(\cos\theta)$ as claimed