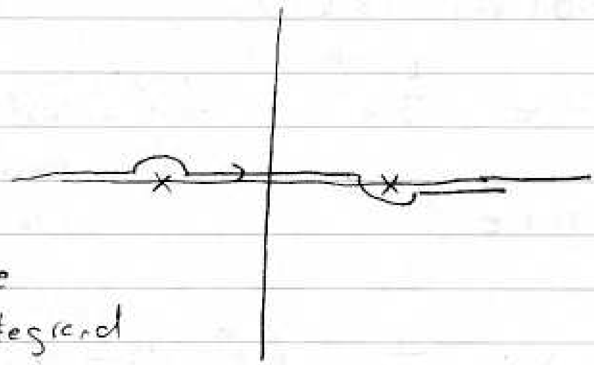
Supplement 5 - Alternative Green's function

To calculate the Green's function for scattering, we had to deform the contour in our integral.

We used



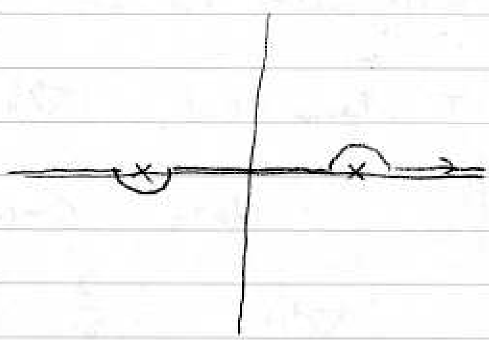
where x's indicate the poles of the integrand

I said that deforming the contour differently would give a different Green's function G' that still solved

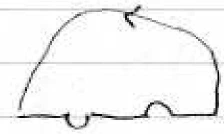
$$(\nabla^2 + k^2) G' = \delta^3(\vec{r})$$

I'll demonstrate that here.

Say we used contour:



For $\int_{-\infty}^{\infty} \frac{s\, e^{isr}}{k^2 - s^2}\, ds$, close in upper half plane:



$$= -\oint \left(\frac{s e^{isr}}{s-k}\right) \frac{1}{s+k}\, ds$$

$$= -2\pi i \, \frac{(-k)\, e^{-ikr}}{(-2k)} = -\pi i\, e^{-ikr}$$

For $\int_{-\infty}^{\infty} \frac{se^{-isr}}{k^2-s^2}\, ds$, close in lower half plane

$$= -\oint \left(\frac{se^{-isr}}{s+k}\right)\frac{1}{s-k}\, ds$$

$$= +2\pi i \, \frac{ke^{-ikr}}{2k}$$

(including minus sign because contour is traverse in clockwise sense)

$$= \pi i e^{-ikr}$$

So $G'(r) = \frac{1}{4\pi^2}\frac{1}{2ir}\left[\int_{-\infty}^{\infty} \frac{se^{isr}}{k^2-s^2}\, ds - \int_{-\infty}^{\infty} \frac{se^{-isr}}{k^2-s^2}\, ds\right]$

$$= \frac{1}{4\pi^2}\frac{1}{2ir}\left[-\pi i e^{-ikr} - \pi i e^{-ikr}\right]$$

$$= -\frac{1}{4\pi}\frac{e^{-ikr}}{r}$$

With other contour, got $G(r) = -\frac{1}{4\pi}\frac{e^{+ikr}}{r}$

I claim that $(\nabla^2+k^2)(G-G') = 0$

Note $G-G' = -\frac{1}{4\pi}\frac{\sin kr}{r} \equiv H(r)$

$$\nabla^2 H = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} H = -\frac{1}{4\pi r^2}\frac{\partial}{\partial r} r^2\left(\frac{k\cos kr}{r} - \frac{\sin kr}{r^2}\right)$$

$$= -\frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left(kr\cos kr - \sin kr\right)$$

$$= -\frac{1}{4\pi r^2}\left(k\cos kr - k^2 r\sin kr - k\cos kr\right)$$

$$= \frac{k^2}{4\pi}\frac{\sin kr}{r} = -k^2 H$$

Thus $(\nabla^2+k^2)H = 0$ as claimed.