Supplement 5 - Alternative Green's Function

To calculate the Green's function for scattering, we had to deform the contour in our integral.

We used

\[ \text{where } \text{x's indicate the poles of the integral} \]

I said that deforming the contour differently would give a different Green's function \( G' \) that still solved

\[(\nabla^2 + k^2) G' = \delta^2(r^2)\]

I'll demonstrate that here.

Say we used contour:

For \( \int \frac{s e^{isr}}{k(s-k)} \, ds \), close in upper half plane:

\[\begin{align*}
&= -\frac{1}{s-k} e^{i kr} \\
&= -2\pi i \frac{e^{-ikr}}{(-2k)} = -\pi i e^{-ikr}
\end{align*}\]
For \( \int_{\gamma} \frac{se^{-isr}}{k^2s^2} \, ds \), close in lower half plane

\[
= -\frac{\pi}{2} \left( \frac{se^{-isr}}{s-k} \right) \, ds
\]

\[
= +2\pi i \frac{ke^{-ikr}}{2k} \quad \text{(including minus sign because contour is traversed in clockwise sense)}
\]

So \( G(r) = \frac{1}{4\pi^3} \frac{1}{2i} \left[ \frac{se^{-isr}}{k^2s^2} ds - \frac{se^{-isr}}{10^2r^2} ds \right] \)

\[
= \frac{1}{4\pi^3} \frac{1}{2i} \left[ -\pi i e^{-ikr} - \pi i e^{-ikr} \right]
\]

\[
= -\frac{1}{4\pi} \frac{e^{-ikr}}{r}
\]

With other contour, \( G(r) = -\frac{1}{4\pi} \frac{e^{+ikr}}{r} \)

I claim that \( (\nabla^2 + k^2)(G-G') = 0 \)

Note \( G-G' = -\frac{1}{4\pi} \frac{\sin kr}{r} = H(r) \)

\[
\nabla^2 H = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} H = -\frac{1}{4\pi^3} \frac{1}{r^2} \left( \frac{k\cos kr - \sin kr}{r^2} \right)
\]

\[
= -\frac{1}{4\pi^3} \frac{1}{r^2} \left( k\cos kr - \sin kr \right)
\]

\[
= -\frac{1}{4\pi^3} \frac{1}{r^2} \left( k\cos kr - k^2 \sin kr - k^2 \sin kr \right)
\]

\[
= \frac{k^2}{4\pi} \frac{\sin kr}{r} = -k^2 H
\]

Thus \( (\nabla^2 + k^2)H = 0 \) as claimed.