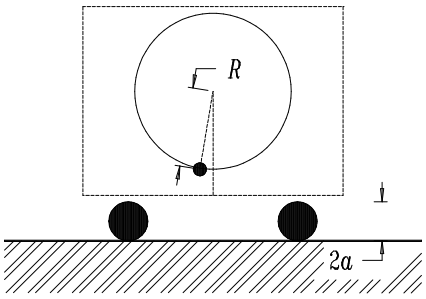
Do all 6 problems. Time limit 3 hrs. Closed Goldstein, open Noble's Notes.

1. A bead of mass m slides freely on a vertical hoop in the shape of a circle of radius R. The hoop is attached to a cart on wheels that is free to roll without slipping on a horizontal surface, as shown to the right. The mass of the hoop and cart is negligible compared with that of the *four* (4) wheels, each of which has mass M and radius a. If the wheels are uniform cylinders,



a) Write down the Lagrangian of the total system.

b) Write the equations of motion for the system.

c) Find any constant(s) of the motion.

d) Find the frequency of small oscillations about any stable equilibria of the bead.

e) Discuss the dependence of this frequency on the wheel radius a.

Solution

a)

$$T_{bead} = \tfrac{1}{2}m\left[\left(R\cos\theta\dot{\theta} + \dot{X}\right)^2 + \left(R\sin\theta\dot{\theta}\right)^2\right]$$
$$= \tfrac{1}{2}mR^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{X}^2 + mR\cos\theta\dot{\theta}\dot{X}$$

$$T_{cart} = \tfrac{1}{2}4M\dot{X}^2 + \tfrac{1}{2}4I\dot{\varphi}^2$$
$$I = \tfrac{1}{2}Ma^2 \qquad \dot{X} = a\dot{\varphi}$$
$$\therefore \quad T_{cart} = 3Ma^2\dot{\varphi}^2$$
$$L = \tfrac{1}{2}(6M+m)a^2\dot{\varphi}^2 + \tfrac{1}{2}mR^2\dot{\theta}^2 + mRa\cos\theta\dot{\theta}\dot{\varphi} + mgR\cos\theta$$

b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = \frac{d}{dt}\left[mRa\cos\theta\dot{\theta} + (6M+m)a^2\dot{\varphi}\right] = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = mR^2\ddot{\theta} + mRa\frac{d}{dt}(\dot{\varphi}\cos\theta) + mR\sin\theta\left(g + a\dot{\varphi}\dot{\theta}\right) = 0$$

c)

$$\frac{\partial L}{\partial \dot{\varphi}} = mRa\cos\theta\dot{\theta} + (6M+m)a^2\dot{\varphi} = \text{constant}$$

$$\dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} - L = H = \text{constant}$$

d) Small oscillations imply we keep only terms of first order in θ:

$$mRa\ddot{\theta} + (6M+m)a^2\ddot{\varphi} \approx 0$$

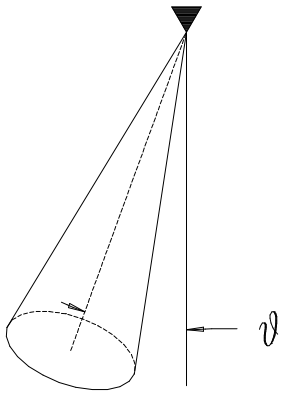
$$mR^2\ddot{\theta} + mRa\ddot{\varphi} + mgR\theta \approx 0$$

$$\ddot{\theta} + \tfrac{6M+m}{6M}\tfrac{g}{R}\theta \approx 0$$

$$\omega^2 = \tfrac{6M+m}{6M}\tfrac{g}{R}$$

e) There is no dependence on a—it all cancels out, even in the exact equations of motion. This is because the only time constant is proportional to the radius of the hoop. Since there is no other acceleration in the problem (that affects the wheels, *e.g.*) there is no time constant that depends on the wheel radius, hence no dependence on the ratio a/R.

2. A pendulum consists of a uniform right circular cone of height h and base radius r pivoted from its apex as shown. Its motion is confined to the x–y plane.



a) Calculate the frequency of small oscillations.

b) For what ratio r/h is this frequency the same as that for a simple pendulum of length h?

Solution

We require the moment of inertia I of the cone about a rotational axis that passes through the apex and is perpendicular to the cone's altitude. We also require the location R of the cone's center of mass, relative to the apex. Once we have these, the frequency of small oscillations is manifestly

$$\omega^2 = \frac{MgR}{I}$$

as for any pendulum. For a cone of uniform density the mass is

$$M = \rho\int_0^h dz\ \pi\, r(z)^2$$

where

$$r(z) = \frac{r}{h}z$$

so that

$$M = \frac{1}{3}\rho\pi r^2 h\ ;$$

also

$$R = \frac{1}{M}\rho\int_0^h dz\ \pi\, r(z)^2\, z = \frac{3}{4}h\ .$$

To calculate the moment of inertia we note that if we slice the cone into disks of mass

$$dm(z) = \rho\ \pi\ r(z)^2\ dz\ ,$$

each disk rotates by θ around a diameter, when the cone rotates by θ about an axis perpendicular to its altitude. Thus, by the parallel axis theorem, the moment of inertia of a disk located a distance z from the apex is

$$dI(z) = dm(z)\, z^2\ +\ \rho\ dz\int_0^{r(z)} db\, b\int_0^{2\pi} d\varphi\, b^2\cos^2\varphi\ =\ dm(z)\left[z^2 + \frac{1}{4}r(z)^2\right].$$

This yields a total moment of inertia

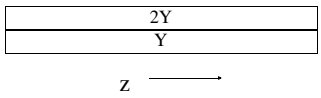
$$I = \int_0^h dI(z) = \frac{1}{5}\rho\ \pi\ \frac{r^2}{h^2}\left[1 + \frac{r^2}{4h^2}\right]h^5\ =\ \frac{3}{5}Mh^2\left[1 + \frac{r^2}{4h^2}\right].$$

The frequency of small oscillations is then given by

$$\omega^2 = \frac{MgR}{I}\ =\ \frac{5}{4}\left(1 + \frac{r^2}{4h^2}\right)^{-1}\ \frac{g}{h}\ .$$

b) When the radius of the base equals the altitude of the cone, the above reduces to $\omega^2 = \frac{g}{h}$.

3. A beam is composed of a laminate of two kinds of wood, whose Young's moduli differ by a factor of two. Taking the bottom of the more flexible piece (Young's modulus Y) to be $x=0$, and taking the z-direction to be horizontal as usual, if the laminations are each of *the same* thickness $h/2$ and the same width w (in the y–direction)



a) find the *vertical* location of the neutral surface;

b) write an expression for the deflection energy corresponding to a deflection function ξ(z) in the x-direction;

c) compare the Young's modulus (for deflections in the x-direction) of the laminate to that of a uniform piece of the more flexible wood, with the same overall dimensions (*i.e.* total height h).

Solution

We recall that the neutral surface is located at a height where the first moment of $Y(x)$ vanishes. Thus

$$0 = \int_0^w dy \int_0^b dz\, Y(x)\left(x - x_{neut}\right)$$

or

$$x_{neut} = \frac{\int_0^b dx\, x\, Y(x)}{\int_0^b dx\, Y(x)} = \left(\frac{Y}{\frac{3Yh}{2}}\right)\int_0^b dx\, x\left[2\theta(x - h/2) + \theta(h/2 - x)\right] = \frac{7}{12}h\,.$$

b) The deflection energy of the laminated piece (assumed length L) is

$$E = \frac{1}{2}\int_0^w dy \int_0^b dx \left(x - x_{neut}\right)^2 Y(x) \int_0^L dz \left(\frac{d^2\xi}{dz^2}\right)^2$$

c) In this case the coefficient of $\int_0^L dz \left(\frac{d^2\xi}{dz^2}\right)^2$ is

$$\frac{Y}{2}wh^3 \cdot \frac{594}{3 \cdot 1728} \equiv \frac{Y}{2}wh^3 \cdot \frac{11}{96}$$

whereas when the object is made from the more flexible wood alone the coefficient is

$$\frac{Y}{2}wh^3 \cdot \frac{1}{12}$$

The effective Young's modulus is therefore

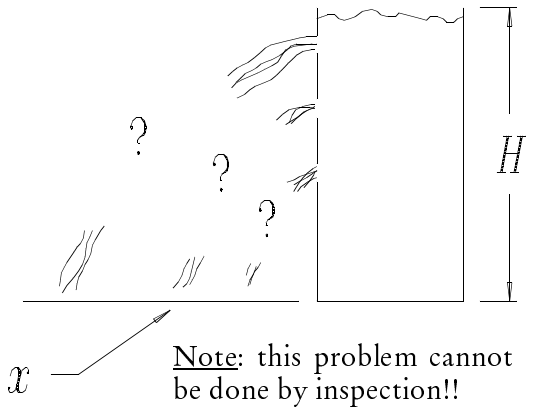
$$Y_{eff} = Y \cdot \frac{11 \cdot 12}{96} = 1.375\, Y\,.$$

4. A vertical container of height H sitting on a table is filled with water, and small holes are punched in the sides, as shown. Water jets out of the holes and hits the table a horizontal distance x from the base of the container.

   a) What is the height of the hole whose jet goes the farthest?

   b) How far does water from *that* hole travel before hitting the table?

(Assume the viscosity can be neglected.)



Note: this problem cannot be done by inspection!!

Solution

Bernouilli's Law for an incompressible fluid is

$$\frac{1}{2}\rho\vec{v}^{\,2} + p = \text{const.}$$

At a height $H - z$ the pressure inside the can is

$$p = \rho g\left(H - z\right) + p_{atm}$$

and the water is not moving. Just outside a hole the pressure is p_{atm} and the water therefore has speed

$$v = \left(\frac{2\rho g\,(H-z)}{\rho}\right)^{1/2} = \left[2g\,(H-z)\right]^{1/2}.$$

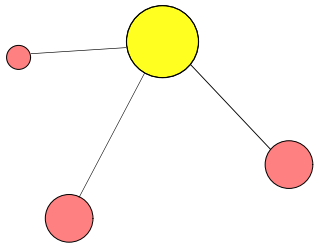
Neglecting air friction, the (horizontal) distance a horizontally-projected object that must simultaneously fall a distance z can travel is therefore

$$x = \left[2g\,(H-z)\right]^{1/2}\cdot\left[\frac{2z}{g}\right]^{1/2} = \left[4\,(H-z)\,z\right]^{1/2}.$$

This is maximum when $z = \frac{H}{2}$; the horizontal distance travelled is then H.

5. Ammonia (NH_3) is a triaxial molecule. Suppose a creature dwelling on a planet with an ammonia atmosphere, at pressure twice that of the Earth, but with the same temperature range (293 ^{o}K), wishes to design a flute whose lowest tone is middle A (440 Hz). How long would the flute be?

Solution



A triaxial molecule has 3 effective rotational degrees of freedom at normal temperatures, hence by the equipartition theorem, a gas of such molecules will have internal energy

$$U = 6\cdot\frac{1}{2}RT = 3RT$$

per mole. Since sound propagation is isentropic (adiabatic!), we have

$$pdV + dU = TdS \equiv 0$$

and since, from the Perfect Gas Law,

$$pdV + Vdp = RdT \equiv \frac{1}{3}dU,$$

we have

$$4pdV + 3Vdp = 0$$

or (since $dV/V \equiv -d\rho/\rho$),

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{4/3}.$$

From the notes on fluids we find that the speed of propagation is

$$u^2 = \gamma p_0/\rho_0 \equiv \gamma RT/\mu$$

where μ is the molar mass.

The speed of sound is then

$$u = u_{air} \times \sqrt{\frac{\gamma_{NH_3}\ \mu_{air}}{\gamma_{air}\ \mu_{NH_3}}}$$

Next we use the fact that the solutions of the wave equation for the flute, corresponding to the lowest mode, have the form (taking the open end of the flute to be $z=0$)

$$\Delta\rho \equiv \eta(r, \varphi, z, t) = A\, e^{i\omega t} \sin kz$$

where

$$k^2 = \frac{\omega^2}{u_s^2}$$

and (because the normal component of velocity must vanish at $z=L$)

$$k \cos kL = 0$$

giving

$$\omega_n \equiv 2\pi \nu_n = u_s \frac{\pi}{L}\left(n + \frac{1}{2}\right).$$

Thus for the fundamental we find

$$\nu_0 = 440\ \mathrm{s}^{-1} = \frac{u_s}{4L}$$

$$L = \frac{u_s}{4\nu_0} = \frac{1.274 \times 343\ \mathrm{m/s}}{4 \times 440\ \mathrm{s}^{-1}} = 24.8\ \mathrm{cm}\ .$$

6. Assuming the Moon were 16 times more massive than it is, and located twice as far from the Earth as it is presently, what would be the effect on the Earth's average deep-water tides?

Solution

From the discussion in class, not to mention the notes on acelerated systems (or you could rederive it), the height of the tide is proportional to the mass of the Moon and inversely proportional to the 3rd power of distance. Hence if the former is multiplied by 16 and the latter by 2, the net result is to double the height of the tide.