Homework #3—solutions

Goldstein p. 66

Problem 11: A particle is placed at the top of a vertical hoop and slides off. Calculate the reaction of the hoop on the particle, and the height at which the particle slides off.

Solution:
The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$$

and the constraint is

$$x^2 + y^2 = r^2 = R^2.$$

Following Goldstein, p. 46ff we see that in terms of virtual displacements δr and $\delta\theta$, we have

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + 2\lambda r = 0,$$

or

$$m\ddot{r} = 2\lambda r + mr\dot{\theta}^2 - mg\cos\theta;$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0,$$

or

$$mr^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = mg\sin\theta.$$

Now, as long as the constraint is satisfied, $\dot{r} = 0$ so that (applying the initial condition $\dot{\theta}_0 = 0$)

$$\frac{1}{2}\dot{\theta}^2 + \frac{g}{R}\cos\theta = \text{constant} = \frac{g}{R}.$$

Then, clearly,

$$m\ddot{r} = f_{constraint} + mr\dot{\theta}^2 - mg\cos\theta = f_{tot} = 0.$$

We then see that the mass leaves the hoop when the constraint force in the radial direction vanishes, *i.e.* when

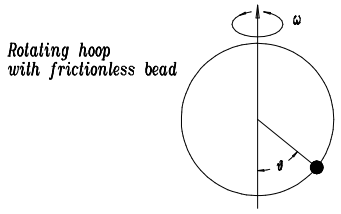
$$f_{constraint} = mg(3\cos\theta - 2) = 0$$

or

$$y = R\cos\theta = \frac{2}{3}R.$$

Since we are measuring from the center of the hoop, this means the height is $5R/3$.

Problem 17: A bead slides frictionlessly on a vertical rotating hoop, as shown to the right. Obtain the Lagrangian and analyze the motion, showing that there is a critical angular velocity ω above which the bead has two equilibrium positions, whereas for smaller angular velocities the only (stable) equilibrium position is the bottom of the hoop.



Rotating hoop with frictionless bead

Solution:
Measuring angles from bottom dead center, as shown, we see that the velocity is

$$R^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right).$$

hence the Lagrangian is

$$L = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgR\cos\theta.$$

The Euler-Lagrange equation is then

$$mR^2\ddot{\theta} = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta,$$

or

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta.$$

The angular acceleration vanishes when

$$\sin\theta = 0$$

or when

$$\cos\theta = \frac{\Omega^2}{\omega^2}, \text{ where } \Omega^2 = \frac{g}{R}.$$

That is, the critical angular velocity is $\omega_{crit} = \Omega$.

Problem 19: A particle of mass m slides on a wedge of mass M and angle a that can slide on a plane. There is no friction between particle and wedge or wedge and plane. Find the equation of motion for particle and wedge, and also find the force of constraint using Lagrange multipliers.

Solution:

Let $X(t)$ be the coordinate of the wedge, and (x, y) those of the particle. Then the constraint is

$$y = (x - X)\tan\alpha .$$

The Lagrangian for the system is

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

and the equations of motion are

$$m\ddot{y} = -mg + \lambda$$
$$m\ddot{x} = -\lambda\tan\alpha$$
$$M\ddot{X} = \lambda\tan\alpha .$$

Since

$$\ddot{y} = \left(\ddot{x} - \ddot{X}\right)\tan\alpha ,$$

we see that

$$-g + \frac{\lambda}{m} = -\lambda\tan^2\alpha\left(\frac{1}{m} + \frac{1}{M}\right)$$

which may be solved easily for λ:

$$\lambda = \frac{gmM\cos^2\alpha}{M + m\sin^2\alpha} .$$

Thus the general solution is

$$y(t) = y_0 + \dot{y}_0 - \frac{1}{2}g\frac{\sin^2\alpha\,(m+M)}{M + m\sin^2\alpha}t^2$$

$$x(t) = x_0 + \dot{x}_0 - \frac{1}{2}\lambda\tan\alpha\, t^2$$

$$X(t) = X_0 + \dot{X}_0 + \frac{1}{2}\lambda\tan\alpha\, t^2 .$$

Finally, the work done on the particle by the constraint force is

$$dW_p = \lambda\dot{y}\,dt - \lambda\tan\alpha\,\dot{x}\,dt = \lambda\,dt\left(\dot{y} - \dot{x}\tan\alpha\right)$$

and that done on the wedge is

$$dW_w = \lambda\tan\alpha\,\dot{X}\,dt .$$

Clearly, the constraint guarantees that

$$dW_p + dW_w \equiv 0 .$$

We may note that the accelerations in the horizontal direction satisfy

$$m\ddot{x} + M\ddot{X} = 0$$

hence total momentum in the x-direction is conserved:

$$m\dot{x}(t) + M\dot{X}(t) = \left(m + M\right)V_{cm} = \text{constant} .$$

Therefore the constants of integration, $\dot{x}_0$ and $\dot{X}_0$ are constrained by

$$m\dot{x}_0 + M\dot{X}_0 = \left(m + M\right)V_{cm} .$$

Similarly, the constants of integration y_0, x_0 and X_0 and the corresponding velocities satisfy

$$y_0 = \left(x_0 - X_0\right)\tan\alpha$$

$$\dot{y}_0 = \left(\dot{x}_0 - \dot{X}_0\right)\tan\alpha .$$

p. 121

Problem 1: Find the equivalent one-dimensional equation of motion of a frictionless bead sliding in a concave-upward paraboloid of revolution, find the condition for the bead to occupy a circular orbit, and find the frequency of small perturbations from circularity.

Solution:

The equation for the paraboloid is $z = \alpha\left(x^2 + y^2\right)$ and the Lagrangian is obviously

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz .$$

This system cries out for cylindrical coordinates ρ φ and z, in terms of which the Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2\right) - mgz ,$$

and the constraint becomes

$$z = \alpha\rho^2 .$$

The φ coordinate is cyclic, hence

$$m\rho^2\dot{\varphi} = J = \text{constant} .$$

Substituting this in the Euler-Lagrange equation for ρ we find (upon introducing the Lagrange multiplier times the constraint coefficient)

$$\ddot{\rho} = \frac{J^2}{m^2\rho^3} - \frac{2\lambda\alpha}{m}\rho .$$

The corresponding equation for z is

$$\ddot{z} = \frac{\lambda}{m} - g ;$$

however, from the constraint equation,

$$\ddot{z} = 2\alpha\left(\dot{\rho}^2 + \rho\ddot{\rho}\right)$$

we may solve for λ/m and insert it into the ρ equation to get

$$\ddot{\rho} = \frac{J^2}{m^2\rho^3} - 2\alpha g\,\rho - 4\alpha^2\left(\rho\dot{\rho}^2 + \rho^2\ddot{\rho}\right).$$

The factor $\dot{\rho}$ immediately lets us integrate the preceding equation to find

$$\frac{1}{2}\dot{\rho}^2 + \frac{J^2}{2m^2\rho^2} + \alpha g\rho^2 + 2\alpha^2\rho^2\dot{\rho}^2 = \frac{E}{m} .$$

The same result of course could have been found directly from the Hamiltonian, which is constant since there is no explicit time-dependence.

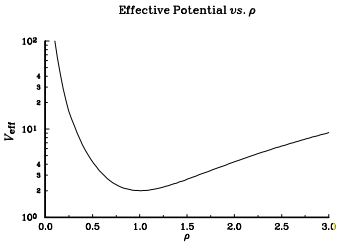
The effective potential

$$V_{eff}(\rho) = \frac{J^2}{2m^2\rho^2} + \alpha g\rho^2$$

has a minimum at

$$\rho_0 = \left[\frac{J^2}{2\alpha m^2 g}\right]^{1/4} ,$$

as shown to the right, so this is the condition for a stable circular orbit with $\ddot{\rho} = \dot{\rho} = 0$.



To calculate the frequency of small oscillations, we substitute $\rho = \rho_0 + \eta(t)$ where η is assumed small, in the equation

$$\ddot{\rho} = \frac{J^2}{m^2\rho^3} - 2\alpha g\,\rho - 4\alpha^2\left(\rho\dot{\rho}^2 + \rho^2\ddot{\rho}\right),$$

getting

$$\ddot{\eta}\left[1 + 4\alpha^2(\rho_0+\eta)^2\right] = \frac{J^2}{m^2(\rho_0+\eta)^3}$$

$$- 2\alpha g\,(\rho_0+\eta) - 4\alpha^2(\rho_0+\eta)\dot{\eta}^2 .$$

Upon eliminating terms of order η^2 and higher, the above reduces to

$$\ddot{\eta}\left[1 + 4\alpha^2\rho_0^2\right] + \left[\frac{3J^2}{m^2\rho_0^4} + 2\alpha g\right]\eta = 0 .$$

Now, if we let

$$J = m\rho_0^2\,\Omega$$

and

$$\alpha g = \overline{\Omega}^2$$

we see that the equation of small oscillations about equilibrium is that of a simple harmonic oscillator,

$$\ddot{\eta} + \omega^2\eta = 0$$

with angular frequency

$$\omega^2 = \frac{3\Omega^2 + 2\overline{\Omega}^2}{1 + 4\alpha^2\rho_0^2} \equiv \frac{4\Omega^2}{1 + 4\alpha^2\rho_0^2} .$$

Problem 3:

The Lagrangian for the system is

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{GMm}{r} ;$$

initially, in an orbit of radius R and period τ

$$\frac{R^3}{\tau^2} = \frac{G(M+m)}{(2\pi)^2}$$

and when the objects are allowed to fall together they obey the equation

$$\dot{r}^2 = 2G(M+m)\left[\frac{1}{r} - \frac{1}{R}\right]$$

or

$$\int_0^R dr \left[\frac{1}{r} - \frac{1}{R}\right]^{-1/2} \equiv R^{3/2} \int_0^1 dx \sqrt{x/(1-x)}$$

$$= R^{3/2} \; \frac{t}{\tau} 2\pi\sqrt{2} \quad .$$

Since

$$\int_0^1 dx \sqrt{x/(1-x)} = \frac{\pi}{2},$$

we see that

$$t = \frac{\tau}{4\sqrt{2}} .$$