Solutions to Homework #4

Goldstein, p. 123:

13. The potential leading to the force

$$\vec{F} = -mC\vec{r} = -\nabla V(r)$$

is obviously

$$V(r) = \frac{1}{2}mCr^2 ;$$

therefore the total effective potential, including the centrifugal potential, is

$$V_{eff}(r) = \frac{-GMm}{r} + \frac{1}{2}mCr^2 + \frac{1}{2}\frac{J^2}{mr^2} .$$

a) The condition for a circular orbit is that the radial acceleration vanish, *i.e.*

$$\frac{-GMm}{r^2} - mCr + \frac{J^2}{mr^3} = 0 .$$

Since the period in an orbit of radius R is

$$\tau = \frac{2\pi}{\dot{\varphi}} = \frac{2\pi\, m\, R^2}{J} = 2\pi\left[\frac{GM}{R^3} + C\right]^{-1/2} ,$$

we see that Kepler's Law is not precisely obeyed.

b) Let $r(t) = R + \eta(t)$; then the potential becomes

$$V_{eff}(r) = V_{eff}(R) + \frac{d}{dr}V_{eff}(R)\,\eta(t) + \frac{1}{2}\frac{d^2}{dr^2}V_{eff}(R)\,\eta^2(t) + \ldots$$

giving the equation of motion for $\eta(t)$

$$\ddot{\eta} + \frac{1}{m}\frac{d^2}{dr^2}V_{eff}(R)\,\eta = 0$$

which is the equation of an harmonic oscillator of period

$$T = 2\pi\left[\frac{GM}{R^3} + 4C\right]^{-1/2} .$$

c) The orbit in the latter case is described by

$$r(t) = R + \eta_0 \cos\left(\frac{2\pi t}{T}\right);$$

since the period T is slightly shorter than the period of the circular orbit, the particle in a perturbed orbit will *advance* slightly, relative to one in an unperturbed orbit.

14. The orbit equation is

$$\frac{du}{d\varphi} = \pm\left[\frac{2m^2}{J^2}\left(-\beta + \frac{k}{m}u\right) - u^2\lambda\right]^{\frac{1}{2}}$$

where

$$\lambda = 1 + \frac{2mh}{J^2},$$

which can be re-expressed as

$$\frac{du}{d\varphi} = \pm\sqrt{\lambda}\left[\left(u_> - u\right)\left(u - u_<\right)\right]^{\frac{1}{2}} .$$

Thus the net change in angle around an orbit is then

$$\Delta\varphi = \frac{2}{\sqrt{\lambda}}\int_{u_<}^{u_>}\frac{du}{\left[\left(u_> - u\right)\left(u - u_<\right)\right]^{1/2}}$$

which gives

$$\Delta\varphi = \frac{2\pi}{\sqrt{\lambda}} \approx 2\pi - \frac{2\pi mh}{J^2} .$$

Obviously the rate of precession is

$$\Omega = \frac{-2\pi mh}{J^2\, T}$$

where T is the period.

Now let us do the same thing in a rotating coordinate system: here we see that if the system is rotating about the z-axis with constant angular velocity $\vec{\Omega} = \Omega\,\hat{z}$, the particle's velocity with respect to space-fixed axes is

$$\vec{v} = \vec{v}' + \vec{\Omega}\times\vec{r}$$

giving

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\left(\dot{\varphi} + \Omega\right)^2\right].$$

This of course leads to equations of motion

$$mr^2\left[\dot{\varphi} + \Omega\right] = J$$

and

$$\ddot{r} = \left(\frac{2h}{m} + \frac{J^2}{m^2}\right)\frac{1}{r^3} - \frac{k}{mr^2};$$

We see that the coefficient of r^{-3} can be set equal to $\frac{J'^2}{m^2} \equiv r^4\dot{\varphi}^2$, *i.e.* the rotation cancels out the effect of the additional r^{-2} potential, if we choose a rotational velocity

$$J' \frac{\Omega}{\dot{\varphi}} \approx -\frac{mh}{J'},$$

which, with $\dot{\varphi} \approx \frac{2\pi}{T}$, gives the previous result.

23. a) $\vec{B} = g\frac{\hat{r}}{r^2}$

$$\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}$$

hence

$$\frac{d}{dt}\left(\vec{r} \times \vec{p}\right) \equiv \vec{r} \times \frac{d}{dt}\vec{p} = q\,\vec{r} \times \left(\frac{\vec{v}}{c} \times \vec{B}\right)$$

$$= \frac{qg}{rc}\left(\vec{v} - \hat{r}\,\vec{v}\cdot\hat{r}\right) \equiv \frac{qg}{c}\frac{d}{dt}\hat{r}$$

and therefore $\frac{d\vec{D}}{dt} = \frac{d}{dt}\left(\vec{L} - \frac{qg}{c}\hat{r}\right) = 0$.

b) Consider $\frac{d}{dt}\left(\vec{p} \times \vec{D}\right) \equiv \vec{F} \times \vec{D}$; after some algebra we get

$$\frac{d}{dt}\left(\vec{p} \times \vec{D}\right) = \left(\frac{qg}{c}\right)^2 \frac{d}{dt}\hat{r}$$

or $\frac{d}{dt}\left[\vec{p} \times \vec{D} - (qg/c)^2\,\hat{r}\right] = 0$.

25. Since the Rutherford scattering formula involves only the square of Q_1Q_2, the relative signs are irrelevant.

27. By straightforward integration of the orbit equation

$$\frac{dr}{d\theta} = \pm r^2\left[\frac{1}{b^2}\left(1 - \frac{V(r)}{E}\right) - \frac{1}{r^2}\right]^{\frac{1}{2}}$$

we find

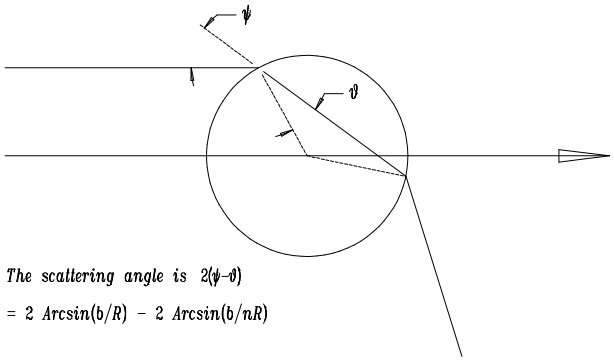
$$\frac{\theta}{2} = -\frac{\pi}{2} + \int_0^{1/R} \frac{du}{\sqrt{b^{-2} - u^2}} + \int_{1/R}^{n/b} \frac{du}{\sqrt{n^2b^{-2} - u^2}}$$

or

$$\frac{\theta}{2} = \sin^{-1}\left(\frac{b}{R}\right) - \sin^{-1}\left(\frac{b}{nR}\right),$$

iff (if and only if) the impact parameter b is less than R. Here $n^2 = 1 + \frac{V_0}{E}$ is the index of refraction.

The figure below shows the geometry of Snell's Law for a sphere.



Now, recalling that

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta}\frac{bdb}{d\theta}$$

and expressing θ in terms of $x = b/R$, we see that

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{1-x^2}\,\sqrt{1-x^2/n^2} + x^2/n$$

which can be solved for x^2:

$$x^2 = \frac{\sin^2\left(\frac{\theta}{2}\right)}{1 + 1/n^2 - \frac{2}{n}\cos\left(\frac{\theta}{2}\right)}.$$

In terms of x we can write

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{\sin\theta}\frac{xdx}{d\theta}$$

or

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{2\sin\theta}\frac{dx^2}{d\theta}.$$

But

$$\frac{dx^2}{d\theta} = \sin\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)\left(1+\frac{1}{n^2}-\frac{2}{n}\cos\left(\frac{\theta}{2}\right)\right)-\frac{1}{n}\sin^2\left(\frac{\theta}{2}\right)\right]\left(1+\frac{1}{n^2}-\frac{2}{n}\cos\left(\frac{\theta}{2}\right)\right)^{-2}$$

and

$$\sin\theta \equiv 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

hence

$$\frac{d\sigma}{d\Omega} = \frac{n^2R^2}{4\cos\left(\frac{\theta}{2}\right)}\frac{\left(n\cos\left(\frac{\theta}{2}\right)-1\right)\left(n-\cos\left(\frac{\theta}{2}\right)\right)}{\left(1+n^2-2n\cos\left(\frac{\theta}{2}\right)\right)^2}.$$