Solutions to Homework # 5

Goldstein, p. 121:

4. The virial theorem in this case proceeds as before except that we express the average kinetic energy in terms of forces rather than potentials:

$$\langle T\rangle = -\frac{1}{2}\sum_k \langle \vec{r}_k \cdot \vec{F}_k \rangle$$

What happens to frictional forces proportional to velocities? If

$$\vec{f}_k \propto \vec{v}_k$$

then

$$\langle \vec{r}_k \cdot \vec{f}_k \rangle \propto \frac{1}{\tau}\int_0^\tau dt\, \vec{r}_k \cdot \vec{v}_k$$

$$\equiv \frac{1}{2\tau}\vec{v}_k^{\,2}\Big|_0^\tau$$

$$\underset{\tau\to\infty}{\to} 0 .$$

Thus

$$\langle T\rangle = -\frac{1}{2}\sum_k \langle \vec{r}_k \cdot \vec{F}_k' \rangle$$

where $\vec{F}_k'$ arise from conservative forces.

9. As we see from the figure below, the meteor follows a hyperbolic orbit; the equation for such an orbit is

$$\frac{1}{r} = \beta\left(1 - \varepsilon\cos\theta\right)$$

where

$$\beta = \frac{GMm^2}{J^2}$$

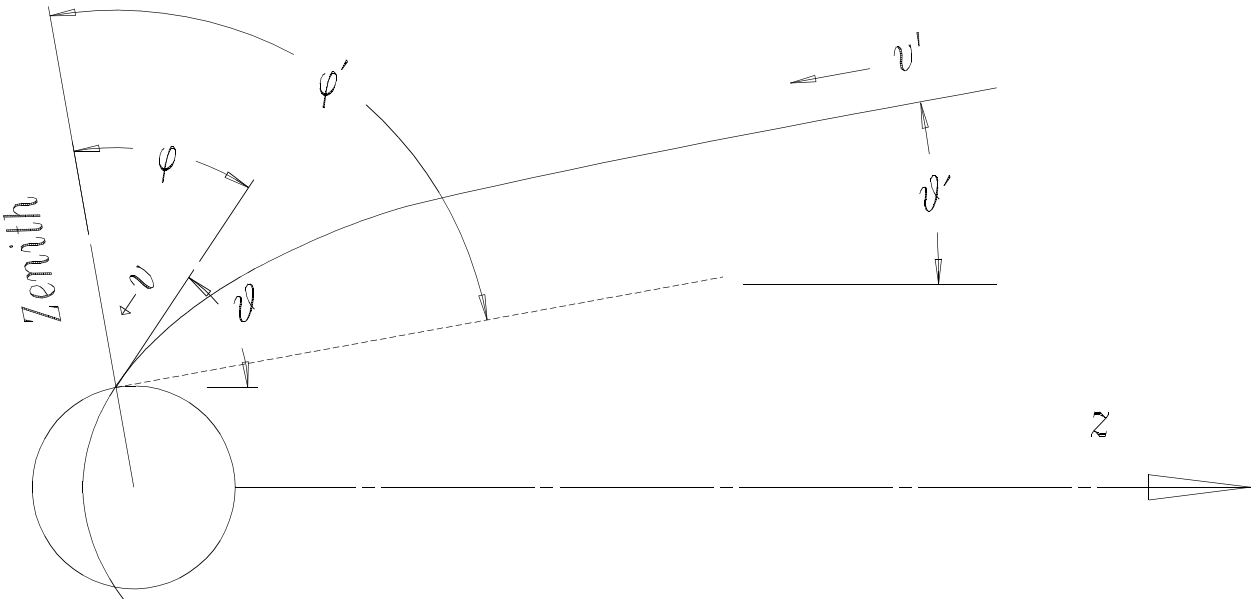
and

$$\varepsilon = \left[1 + \frac{m^2 v'^2 J^2}{g^2 R^4 m^4}\right]^{1/2} .$$

Now, conservation of energy gives

$$v'^2 = v^2 - \frac{2GM}{R} = v^2 - 2gR ,$$

where both v and gR are known quantities. We now have to determine the asymptotic angle φ' with the zenith. To do this we note that the angular momentum of the incoming particle is known:



$$J = |\vec{r}\times\vec{p}| = mv'b \equiv mvR\sin\varphi\,.$$

It is the latter relation we shall use here. Now, from the diagram it is obvious that

$$\theta' = \cos^{-1}\left(\frac{1}{\varepsilon}\right)$$

and

$$\theta = \cos^{-1}\left(\frac{1-1/R\beta}{\varepsilon}\right)$$

Also it is clear from the diagram that

$$\varphi' + \theta' = \varphi + \theta\,;$$

hence

$$\varphi' = \varphi + \cos^{-1}\left(\frac{1-1/R\beta}{\varepsilon}\right) - \cos^{-1}\left(\frac{1}{\varepsilon}\right).$$

With the value $J = mvR\sin\varphi$ we then rewrite

$$\beta = \frac{GMm^2}{J^2} = \frac{g}{v^2\sin^2\varphi}$$

and

$$\varepsilon = \left[1 + \frac{v'^2v^2\sin^2\varphi}{g^2R^2}\right]^{1/2}.$$

Since all the quantities in the expression for φ' are now known, we have solved the problem.

Extra problems:

1. Calculate the heights of Lunar and Solar tides. The expression we derived in class for the height of the tide as a function of angle relative to the direction of the Moon (or Sun) was

$$h(\theta) = R_E\frac{M_2}{M_E}\left(\frac{R_E}{\rho}\right)^3 P_2(cos\theta)\,.$$

Here ρ is the distance from Earth to the attracting body and θ is the (polar) angle from some point on the Earth's surface to the axis defined by the centers of mass of Earth and the attracting body.

The Legendre polynomial of second order evaluates to 1 at $\theta=0$ and π, and to $-\frac{1}{2}$ at $\theta=\pi/2$. Thus the maximum difference between high and low tides is

$$\Delta h = \frac{3}{2}R_E\frac{M_2}{M_E}\left(\frac{R_E}{\rho}\right)^3.$$

For the Earth-Moon system Δh is 0.55 m.

For the Solar tide the maximum difference is 0.25 m, since the mass of the earth is about 6×10^{24} kg and that of the Sun 2×10^{30} kg.

Thus it is very important whether the Moon and Sun and Earth are aligned ("spring" tide) or the Moon and Sun are at 90° ("neap" tide). Spring tides occur when the Moon is *full* or *new*, neap tides at the ¼ and ¾ phases.

2. Using the General Relativistic "Lagrangian" of a test particle in the field of a gravitating mass (p. 54 of the notes) in conjunction with the small-angle scattering approximation, derive Einstein's result for the deflection of light by a gravitating body.

Solution:

Since $-mc^2$ is a constant factor we drop it, taking

$$L = \left[1 - 2\Phi - \frac{\dot{r}^2/c^2}{1-2\Phi} - \frac{r^2\dot{\varphi}^2}{c^2}\right]^{1/2},$$

where the (dimensionless) gravitational potential is

$$\Phi = \frac{GM}{rc^2}\,.$$

Note that at the Sun's surface, $\Phi \approx 2\times10^{-6}$.

Normally v^2/c^2 and Φ are of the same order of magnitude; however, here we will have

$$\dot{x} = v_x \approx c\,,\quad |\dot{y}| = |v_y| << c\,.$$

We therefore need to be careful with our approximations. We note first that because φ is a cyclic coordinate,

$$\frac{d}{dt}\left[\frac{r^2\dot{\varphi}}{L}\right] = J = \text{constant}\,;$$

moreover, when L appears in denominators we may replace it by

$$L \approx \sqrt{1 - v^2/c^2} \equiv \gamma^{-1},$$

since we shall be keeping only terms of *first* order in Φ.

We thus rewrite the Lagrangian as

$$L \approx \left[1 - 2\Phi \; - \; \frac{v_x^2 + v_y^2}{c^2} (1 + 2\Phi) \; + \right.$$

$$\left. + \; 2\Phi \frac{J^2}{r^2 c^2 \gamma^2} \right]^{1/2}$$

and keeping $v_x \approx$ constant, find

$$\gamma \; \frac{dv_y}{dt} \approx \gamma \, c^2 \left[(1 + v^2/c^2) \frac{\partial \Phi}{\partial y} \; - \right.$$

$$\left. - \; (1 - v^2/c^2) \frac{\partial}{\partial y} \left(\frac{\Phi(r) \, J^2}{r^2 c^2} \right) \right]$$

Keeping in mind that

$$x \approx v_x \, t$$

and that

$$\Delta\theta \approx \frac{v_y(t \to \infty)}{v_x},$$

we translate the above into

$$\Delta\theta \approx \frac{c^2}{v_x^2} \int_{-\infty}^{\infty} dx \left[(1 + v^2/c^2) \frac{\partial \Phi}{\partial y} \; - \right.$$

$$\left. - \; J^2 \, (1 - v^2/c^2) \frac{\partial}{\partial y} \left(\frac{\Phi}{r^2 c^2} \right) \right]$$

In the limit as $v_x \approx v \to c$, the first term in the integral goes to Einstein's result,

$$\Delta\theta \Big|_{GR} = 2 \; \Delta\theta \Big|_{Newton}$$

whereas the second term goes to 0.