Solutions to HW 6

Goldstein, p. 187:

22. In the rotating frame the equation of motion is

$$\frac{d\vec{v}}{dt} \approx -\vec{g} + 2\vec{v}\times\vec{\Omega},$$

where we ignore the term quadratic in $\vec{\Omega}$,

$$m\vec{\Omega}\times\left(\vec{x}\times\vec{\Omega}\right).$$

Now, set up a local Cartesian coordinate system $\hat{r}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ at the surface of the Earth, keeping in mind that $\vec{\Omega} = \Omega\,\hat{z}$, where

$$\hat{z} = \hat{r}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta,$$

and θ is the co-latitude.

Then

$$\dot{v}_r = -g + 2\Omega\, v_\varphi \sin\theta$$
$$\dot{v}_\theta = \quad 2\Omega\, v_\varphi \cos\theta$$
$$\dot{v}_\varphi = \quad -2\Omega\,(\, v_\theta \cos\theta + v_r \sin\theta\,)$$

Eliminating v_r and v_θ we have

$$\ddot{x}_\varphi + 4\Omega^2 x_\varphi = 2\Omega\sin\theta\left[\, gt - v_r(0)\,\right],$$

and recalling that we have already dropped quadratic terms in Ω, we find

$$x_\varphi \approx 2\Omega\sin\theta\left[\,\frac{gt^3}{6} - \frac{v_r^0\, t^2}{2}\,\right].$$

Since the maximum height in the first case is

$$h = \frac{(v_r^0)^2}{2g}$$

and the times of flight are, respectively,

$$t_1 = 2\left(\frac{2h}{g}\right)^{1/2},\quad t_2 = \left(\frac{2h}{g}\right)^{1/2},$$

we easily see that the net deflections are

$$x_\varphi^{(1)} = 2h\,\Omega\sin\theta\left(\frac{2h}{g}\right)^{1/2}\left(\frac{-4}{3}\right)$$

$$x_\varphi^{(2)} = 2h\,\Omega\sin\theta\left(\frac{2h}{g}\right)^{1/2}\left(\frac{1}{3}\right)$$

i.e., $x_\varphi^{(1)} = -4x_\varphi^{(2)}$.

23. From our preceding equations, neglecting motion in the $\hat{r}$ direction, we have

$$\dot{v}_\theta = \quad 2\Omega\, v_\varphi \cos\theta$$
$$\dot{v}_\varphi = -2\Omega\, v_\theta \cos\theta$$

hence

$$x_\theta = \quad \Omega\, v_\varphi \cos\theta\, t^2 + v_\theta\, t$$
$$x_\varphi = -\Omega\, v_\theta \cos\theta\, t^2 + v_\varphi\, t$$

and so

$$\Delta\psi \approx \frac{|\,\Delta\vec{x}\,|}{|\vec{x}|} = \frac{\Omega\cos\theta\, vt^2}{vt} = \Omega\cos\theta\,.$$

24. With the definitions of a local Cartesian coordinate system (z is vertical) the equations of motion for a pendulum of length λ are

$$\ddot{x} + \omega^2 x + 2\Omega\cos\theta\,\dot{y} \approx 0$$
$$\ddot{y} + \omega^2 y - 2\Omega\cos\theta\,\dot{x} \approx 0$$

where $\omega^2 = \frac{g}{\lambda}$, and we have employed the small-angle approximation. Obviously, in a coordinate system rotating about the local z-axis with angular velocity $\dot{\chi} = -\Omega\cos\theta$, the term in Ω would disappear, so we must conclude that the system is precessing at a rate of

$$2\pi\cos\theta/\text{day}\,.$$

We couldalso find this by solving explicitly for $x(t)$ and $y(t)$, assuming that $\Omega << \omega$.

Goldstein, p. 240:

14. By inspection we see that the length of the rod is $r\sqrt{3}$, and its height is $r/2$. From the parallel axis theorem, then, the moment of inertia is the principal moment about the rod's center plus the mass times the squared distance to the center of rotation:

$$I = m\frac{\left(r3^{1/2}\right)^2}{12} + m\left(\frac{r}{4}\right)^2 = \frac{mr^2}{2}.$$

The potential energy is

$$V = \frac{-mgr}{2}\cos\theta$$

so the equation of motion is

$$I\ddot{\theta} + \frac{mgr}{2}\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{g}{r}\sin\theta = 0.$$

As was to be shown, this is the equation for a simple pendulum of length r.

15. The acceleration is constant, so the problem is that of a pendulum in a gravitational field $-f$. The equation of motion is then

$$\ddot{\theta} + \frac{mfa}{I}\sin\theta = 0.$$

The time for a pendulum to fall by $\pi/2$ is

$$t = \frac{1}{\omega\sqrt{2}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}},$$

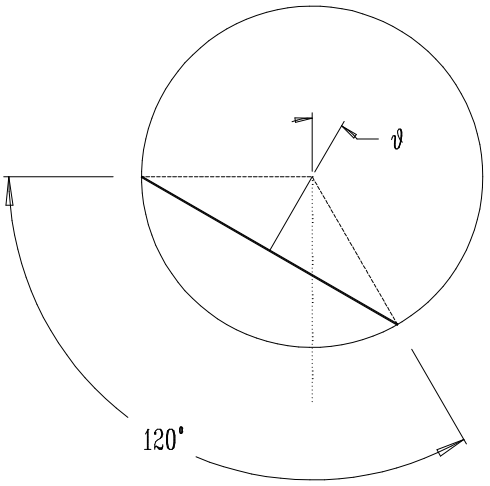
where

$$\omega^2 = \frac{fa}{r_0^2};$$

using the transformation $v = \cos^2\theta$ we can re-write this as

$$t = \frac{1}{\omega\sqrt{2}}\int_0^1 dv\, v^{1/4-1}\left(1-v\right)^{1/2-1}$$

$$= \frac{B(1/4,1/2)}{2\omega\sqrt{2}} \approx \frac{1.854}{\omega}.$$



Now, the radius of gyration is $r_0^2 = \frac{w^2}{3}$ and the distance of the cm from the hinge is 2 ft, hence the angular frequency is

$$\omega = \sqrt{0.375}\ \text{sec}^{-1}$$

and the time to fall is

$$t \approx \frac{1.854}{\sqrt{0.375}} = 3.027\ \text{sec}.$$