Solutions to Homework # 7

Goldstein, p. 271:

1. The kinetic and potential energies are

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$V = mgy$$

$$y = ax^4 .$$

The equation of motion is

$$\ddot{x}\left(1 + 16a^2x^6\right) + 48a^2\dot{x}^2x^5 + 4gax^3 = 0,$$

which implies x=0 is the equilibrium point, in agreement with common sense. Manifestly this is a situation for which there is no quadratic expansion of the potential about the equilibrium, and the motion is therefore not harmonic. For small amplitudes, we neglect terms of higher order than x in the kinetic energy, giving

$$\ddot{x} + 4gax^3 \approx 0 .$$

It is easy to see the period of small oscillations is

$$T = \frac{4}{x_{\max}}\left[\frac{\pi}{2ga}\right]^{1/2}$$

where $x_{\max}$ is the initial displacement of the mass. As in all graviational problems, the period is independent of the mass of the object.

Extra problem:

A particle of mass m moves 3-dimensionally in the potential

$$V(r) = \frac{a}{r^4} - \frac{b}{r^2} .$$

a) Find the condition for a stable circular orbit.
b) Are small perturbations of a stable circular orbit stable?
c) Express the vibrational frequency of such perturbations in terms of the orbital period.

Solution:

Clearly,

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) - V(r)$$

and we may set $\dot{\theta} = \ddot{\theta} = 0$, $\theta = \frac{\pi}{2}$. Then, as usual the orbital equations become

$$mr^2\dot{\varphi} = J = \text{const.}$$

$$m\ddot{r} = \frac{J^2}{mr^3} - \frac{2b}{r^3} + 4\frac{a}{r^5}$$

Unless $\frac{J^2}{m} < 2b$, $\ddot{r} > 0$ and the particle will be continuously accelerated outward. That is, the (repulsive) centrifugal potential cannot exceed in magnitude the attractive $1/r^2$ potential.

The equilibrium radius is then

$$r_0^2 = \frac{4a}{2b - J^2/m} ;$$

letting $r(t) = r_0 + \chi(t)$ we have

$$m\ddot{\chi} = \frac{\left(J^2/m - 2b\right)}{\left(r_0 + \chi\right)^3} + \frac{4a}{\left(r_0 + \chi\right)^5}$$

or

$$m\ddot{\chi} + \frac{2\left(2b - J^2/m\right)}{r_0^4}\chi \approx 0,$$

i.e. the oscillations are stable when the orbit is.

Finally, the frequency is clearly

$$\omega = \left[\frac{(2b - J^2/m)^3}{8a^2m}\right]^{\frac{1}{2}} .$$