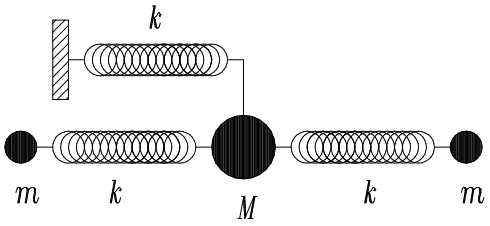
Homework 8

Due: 28 October 1997

Goldstein, p.271

4. The situation is as shown to the right: the kinetic and potential energies are, respectively



$$T = \frac{1}{2}\left(m\dot{x}_1^2 + M\dot{x}_1^2 + m\dot{x}_3^2 \right),$$

$$V = \frac{k}{2}\left(|x_2 - x_1| - b \right)^2 + \frac{k}{2}\left(x_2 \right)^2 + \frac{k}{2}\left(|x_2 - x_3| - b \right)^2$$

so that the Lagrangian can be written

$$L \approx \frac{1}{2}\left(m\dot{\xi}_1^2 + M\dot{\xi}_1^2 + m\dot{\xi}_3^2 \right) - \frac{k}{2}\left((\xi_2 - \xi_1)^2 + \xi_2^2 + (\xi_3 - \xi_2)^2 \right)$$

in terms of the small displacements $\xi_n(t)$.

The normal mode problem then becomes

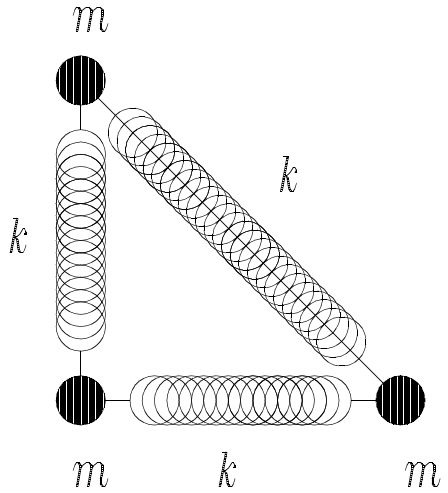
$$\det \begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 3k - M\omega^2 & -k \\ 0 & 0 & k - m\omega^2 \end{pmatrix} = 0$$

or

$$\omega^2 = \Omega^2 \left\{ \frac{1}{2}\left(1 + \frac{3m}{M} \right) \pm \frac{1}{2}\left[\left(1 + \frac{3m}{M} \right)^2 - \frac{4m}{M} \right]^{1/2} \right\}$$

where

$$\Omega^2 = \frac{k}{m}.$$



8. The key here is the potential energy: let

$$\vec{r}(t)_n = \vec{\rho}(t)_n + \vec{r}_n^{(0)} \quad ;$$

then the potentials take the form

$$V_{ij} = \frac{k}{2}\left(|\vec{r}_i - \vec{r}_j| - |\vec{r}_i^{\,(0)} - \vec{r}_j^{\,(0)}| \right)^2 \approx \frac{k}{2}\left[\vec{R}_{ij} \cdot (\vec{\rho}_i - \vec{\rho}_j) \right]^2.$$

where

$$\vec{R}_{ij} = \frac{\vec{r}_i^{\,(0)} - \vec{r}_j^{\,(0)}}{|\vec{r}_i^{\,(0)} - \vec{r}_j^{\,(0)}|}.$$

With

$\vec{r}_1^{\,(0)} = (0, 0)$

$\vec{r}_2^{\,(0)} = (a, 0)$

and

$\vec{r}_3^{\,(0)} = (0, a)$

we find, in terms of the components of the 2-dimensional vectors $\vec{\rho} \equiv (\xi, \eta)$

$$V = \frac{k}{2}\left[\left(\xi_1 - \xi_2 \right)^2 + \frac{1}{2}\left(\eta_3 - \eta_2 + \xi_2 - \xi_3 \right)^2 + \left(\eta_3 - \eta_1 \right)^2 \right]$$

and

$$T = \frac{m}{2}\left[\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2 + \dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2 \right].$$

The corresponding equations for normal modes are

$$\begin{pmatrix} k-m\omega^2 & 0 & -k & 0 & 0 & 0 \\ 0 & k-m\omega^2 & 0 & 0 & 0 & -k \\ -k & 0 & 3k/2-m\omega^2 & -k/2 & -k/2 & k/2 \\ 0 & 0 & -k/2 & k/2-m\omega^2 & k/2 & -k/2 \\ 0 & 0 & -k/2 & k/2 & k/2-m\omega^2 & -k/2 \\ 0 & -k & k/2 & -k/2 & -k/2 & 3k/2-m\omega^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \\ \xi_3 \\ \eta_3 \end{pmatrix} = 0$$

The determinant of the above equation can be rewritten in a more obviously symmetric form:

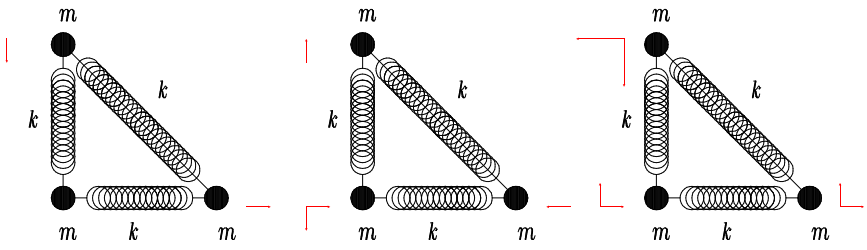
$$\left\| \begin{matrix} \Omega^2-\omega^2 & 0 & -\Omega^2 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\Omega^2-\omega^2 & -\frac{1}{2}\Omega^2 & -\frac{1}{2}\Omega^2 & \frac{1}{2}\Omega^2 & 0 \\ -\Omega^2 & -\frac{1}{2}\Omega^2 & \frac{3}{2}\Omega^2-\omega^2 & \frac{1}{2}\Omega^2 & -\frac{1}{2}\Omega^2 & 0 \\ 0 & -\frac{1}{2}\Omega^2 & \frac{1}{2}\Omega^2 & \frac{3}{2}\Omega^2-\omega^2 & -\frac{1}{2}\Omega^2 & -\Omega^2 \\ 0 & \frac{1}{2}\Omega^2 & -\frac{1}{2}\Omega^2 & -\frac{1}{2}\Omega^2 & \frac{1}{2}\Omega^2-\omega^2 & 0 \\ 0 & 0 & 0 & -\Omega^2 & 0 & \Omega^2-\omega^2 \end{matrix} \right\| = 0$$

from which it is easy to see, using standard manipulations, that

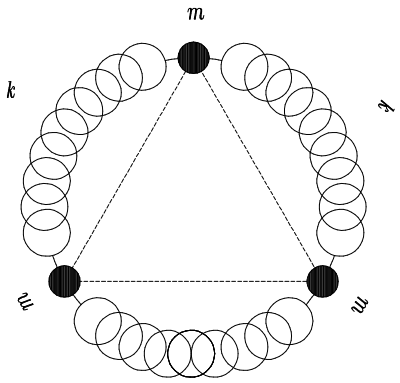
$$\left(\omega^2\right)^3\left(\Omega^2-\omega^2\right)\left(2\Omega^2-\omega^2\right)\left(3\Omega^2-\omega^2\right)=0\,;$$

that is, there is the expected triple zero from rotational and translational symmetry, as well as the three normal modes with frequencies $\omega = (\Omega, \sqrt{2}\Omega, \sqrt{3}\Omega)$. The corresponding eigenvectors are

$$\omega=\Omega:\begin{pmatrix}0\\0\\1\\0\\0\\-1\end{pmatrix};\quad \omega=\sqrt{2}\Omega:\begin{pmatrix}1\\-1\\-1\\0\\0\\1\end{pmatrix};\quad \omega=\sqrt{3}\Omega:\begin{pmatrix}1\\1\\1\\1\\-2\\-2\end{pmatrix}$$



10. The system is as shown to the right. If the angles by which the 3 masses deviate from equilibrium are $\theta_{1,2,3}$ we have



$$T=\frac{R^2}{2}m\left(\dot\theta_1^2+\dot\theta_2^2+\dot\theta_3^2\right)$$

and

$$V=\frac{kR^2}{2}\left[\left(\theta_1-\theta_2\right)^2+\left(\theta_2-\theta_3\right)^2+\left(\theta_3-\theta_1\right)^2\right].$$

Hence the eigenvalue equation to be solved is

$$\begin{pmatrix}2\Omega^2-\omega^2 & -\Omega^2 & -\Omega^2\\ -\Omega^2 & 2\Omega^2-\omega^2 & -\Omega^2\\ -\Omega^2 & -\Omega^2 & 2\Omega^2-\omega^2\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}=0\,,$$

where

$$\Omega^2=\frac{k}{m}\,.$$

The determinant is $D = \omega^2\left(\omega^2 - 3\Omega^2\right)^2$, giving one double and one single root. The corresponding eigenvectors are

$$\omega^2 = 0: \begin{pmatrix}1\\1\\1\end{pmatrix},\ \omega^2 = 3\Omega^2: \begin{pmatrix}1\\-x\\-1+x\end{pmatrix} \text{ or } \begin{pmatrix}-1+x\\-x\\1\end{pmatrix}.$$

Note the degeneracy.

Now looking at perturbations of the spring constant or mass matrices, we have first,

$$\begin{vmatrix} 2\Omega^2+\delta\Omega^2-\omega^2 & -\Omega^2 & -\Omega^2-\delta\Omega^2 \\ -\Omega^2 & 2\Omega^2-\omega^2 & -\Omega^2 \\ -\Omega^2-\delta\Omega^2 & -\Omega^2 & 2\Omega^2+\delta\Omega^2-\omega^2 \end{vmatrix} = 0,$$

whose roots are $\omega^2 = (0,\ 3\Omega^2,\ 3\Omega^2 + 2\delta\Omega^2)$ where $\delta\Omega^2 = \dfrac{\delta k}{m}$. Increasing a spring constant *raises* the frequency of one normal mode but leaves the other two unchanged.

Similarly, for perturbations of one of the masses we have

$$\begin{vmatrix} 2\Omega^2-\omega^2(1+\varepsilon) & -\Omega^2 & -\Omega^2 \\ -\Omega^2 & 2\Omega^2-\omega^2 & -\Omega^2 \\ -\Omega^2 & -\Omega^2 & 2\Omega^2-\omega^2 \end{vmatrix} = 0,$$

whose roots are $\omega^2 = (0,\ 3\Omega^2,\ 3\Omega^2 - 2\Omega^2\,\varepsilon)$, where $\varepsilon = \dfrac{\delta m}{m}$. Increasing one mass lowers the frequency of the corresponding normal mode.

14. From Kirckhoff's Law, we have

$$L_1\ddot{I}_1 + C_1^{-1}I_1 + L_3\left(\ddot{I}_1 - \ddot{I}_2\right) + C_3^{-1}\left(I_1 - I_2\right) = 0$$

$$L_2\ddot{I}_2 + C_2^{-1}I_2 - L_3\left(\ddot{I}_1 - \ddot{I}_2\right) - C_3^{-1}\left(I_1 - I_2\right) = 0,$$

leading to

$$\begin{vmatrix} C_1^{-1}+C_3^{-1} - \omega^2(L_1+L_3) & -C_3^{-1} + \omega^2 L_3 \\ -C_3^{-1} + \omega^2 L_3 & C_2^{-1}+C_3^{-1} - \omega^2(L_2+L_3) \end{vmatrix} = 0$$

This is rather ugly, but it is only quadratic so it can be solved easily.

Extra problem

Show by direct substitution in the original equation

$$m\ddot{x} + \gamma m\dot{x} + kx = f(t)\,,$$

that the particular solution

$$x(t) = \int^{t} d\tau e^{-\gamma/2(t-\tau)} \sin\Omega(t-\tau)\ \frac{f(\tau)}{m\Omega}$$

derived by operator calculus *is* in fact a solution.

Solution

$$x(t) = \mathrm{Im} \int^{t} d\tau\, e^{i\lambda(t-\tau)}\, Q(\tau)$$

where

$$\lambda = \Omega + i\frac{\gamma}{2} \equiv \sqrt{\omega^2 - \gamma^2/4}\ +\ i\frac{\gamma}{2}$$

and

$$Q(t) = \frac{f(t)}{m\Omega}\,.$$

Now, it is easy to see by direct differentiation with respect to t that

$$\dot{x}(t) = \mathrm{Im}\left(i\lambda \int^{t} d\tau\, e^{i\lambda(t-\tau)}\, Q(\tau) \right) + \mathrm{Im}Q(t)$$

but since $Q(t)$ is real, the last term vanishes. If we differentiate again we find

$$\ddot{x}(t) = \mathrm{Im}\left(-\lambda^2 \int^{t} d\tau\, e^{i\lambda(t-\tau)}\, Q(\tau) \right) + Q(t)\ \mathrm{Im}(i\lambda)$$

so that

$$\ddot{x} + \gamma\dot{x} + \omega^2 x =$$

$$\frac{f(t)}{m} +\ \mathrm{Im}\left((\omega^2 + i\lambda\gamma - \lambda^2) \int^{t} d\tau\, e^{i\lambda(t-\tau)}\, Q(\tau) \right).$$

But

$$\omega^2 + i\lambda\gamma - \lambda^2 = \omega^2 + i\Omega\gamma - \frac{\gamma^2}{2} - \Omega^2 - i\Omega\gamma + \frac{\gamma^2}{4}$$

$$\equiv \omega^2 - \Omega^2 - \frac{\gamma^2}{4} \equiv 0\,,$$

hence

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = \frac{f(t)}{m}\,,\ \mathrm{QED}.$$