Homework # 9

Goldstein, p. 597:

1. The derivation of the Lagrangian of a string, by going from a discrete to a continuous system was done explicitly in the notes "Introduction to continuous systems". Therefore I will not repeat it here.

   However, the method of derivation suggested by part b) of the question is interesting. The kinetic energy is trivial: if the transverse displacement at a point x is $\psi(x\,t)$, then the kinetic energy of the section of string between x and $x+dx$ is

$$d(K) = \frac{1}{2}\,\mu(x)\,dx\left(\frac{\partial\psi}{\partial t}\right)^2$$

   or

$$K = \int_0^L \frac{1}{2}\,\mu(x)\left(\frac{\partial\psi}{\partial t}\right)^2 dx\,.$$

   The potential energy is given by the work done in stretching the string: that is,

$$\begin{aligned} d(V) &= \tau\left(ds - dx\right) \\ &\equiv \tau\left(\sqrt{dx^2 + d\psi^2} - dx\right) \\ &= \tau\left(\sqrt{1 + (\partial\psi/\partial x)^2} - 1\right)dx \\ &\approx \frac{1}{2}\,\tau\left(\frac{\partial\psi}{\partial x}\right)^2 dx\,. \end{aligned}$$

   Thus

$$V = \int_0^L \frac{1}{2}\,\tau\left(\frac{\partial\psi}{\partial x}\right)^2 dx\,.$$

2. We have already derived the Lagrangian for sound waves in a gas:

$$L = \frac{1}{2}\int d^3x\left[\bar{\rho}\left(\frac{\partial}{\partial t}\,\vec{\zeta}\,(\vec{x},t)\right)^2 - \bar{\gamma p}\left(\nabla\cdot\vec{\zeta}\,(\vec{x},t)\right)^2\right];$$

   manifestly the Hamiltonian is

$$H = \frac{1}{2}\int d^3x\left[\frac{1}{\bar{\rho}}\left(\vec{\pi}\,(\vec{x},t)\right)^2 + \bar{\gamma p}\left(\nabla\cdot\vec{\zeta}\,(\vec{x},t)\right)^2\right].$$

   We see that the Hamiltonian equations of motion are

$$\frac{\partial\vec{\zeta}}{\partial t} = \frac{\partial\mathscr{H}}{\partial\vec{\pi}} \equiv \frac{1}{\bar{\rho}}\,\vec{\pi}$$

   and

$$\frac{\partial\vec{\pi}}{\partial t} = -\,\frac{\partial\mathscr{H}}{\partial\vec{\zeta}} \equiv \bar{\gamma p}\,\nabla^2\,\vec{\zeta}\,.$$

   In the momentum representation,

$$\vec{\zeta}(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\vec{\eta}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\vec{\pi}(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\vec{\theta}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}};$$

   hence

$$H = \frac{1}{2}\sum_{\mathbf{k}}\left[\frac{1}{\bar{\rho}}\left(\frac{d}{dt}\,\vec{\theta}_{\mathbf{k}}(t)\right)^2 + \bar{\gamma p}\left(\mathbf{k}\cdot\vec{\eta}_{\mathbf{k}}(t)\right)^2\right].$$

4. If the Lagrangian density is (note the incorrect sign in Goldstein)

$$\mathscr{L} = \frac{-1}{2m}\,\nabla\psi^*\cdot\nabla\psi - V\psi^*\,\psi - \frac{1}{2i}\left(\psi^*\,\dot{\psi} - \dot{\psi}^*\,\psi\right)$$

then

$$\frac{\partial \mathscr{L}}{\partial \psi^*} = \frac{1}{2m}\nabla^2\psi - V\psi - \frac{1}{2i}\dot{\psi}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\psi}^*}\right) = \frac{1}{2i}\dot{\psi} \ ;$$

setting these equal yields the Schroedinger equation for ψ:

$$\frac{-1}{2m}\nabla^2\psi + V\psi = -i\dot{\psi} \, .$$

Here of course we have used units with $\hbar = 1$. Clearly the canonical momentum corresponding to ψ^* is

$$\pi^* = \frac{\partial \mathscr{L}}{\partial \dot{\psi}^*} = \frac{1}{2i}\psi \ ;$$

that corresponding to ψ can be obtained from the above equation by complex conjugation.

Manifestly,

$$\mathscr{H} = \dot{\psi}^* \frac{\partial \mathscr{L}}{\partial \dot{\psi}^*} + \dot{\psi}\frac{\partial \mathscr{L}}{\partial \dot{\psi}} - \mathscr{L}$$

giving

$$\mathscr{H} = \frac{1}{2m}\nabla\psi^* \cdot \nabla\psi + V\psi^*\psi \, ,$$

as we might have expected *a priori*.

Extra problem:

Work out the dynamical stabilization equations for a horizontally driven pendulum.

Solution

From p. 75 of the notes, the Euler-Lagrange equation is

$$\ddot{\theta} + \omega^2 \sin\theta + \ddot{X}\cos\theta = 0 \, .$$

Letting

$$\theta = \varphi + \eta$$

as before, we find

$$\ddot{\varphi} + \ddot{\eta} + \omega^2\left(\sin\varphi + \eta\cos\varphi\right) + \frac{\ddot{X}}{r}\left(\cos\varphi - \eta\sin\varphi\right) = 0 \, .$$

As previously, we find

$$\eta \approx \frac{-X}{r}\cos\varphi \, ,$$

$$\ddot{\varphi} + \sin\varphi\left(\omega^2 + \frac{1}{r^2}\langle X\ddot{X}\rangle\cos\varphi\right) \approx 0 \, .$$

If the horizontal driving force is sinusoidal,

$$X(t) = A\cos(\Omega t)$$

the time-average term is

$$\frac{1}{r^2}\langle X\ddot{X}\rangle = \frac{-\Omega^2 A^2}{2r^2} = -\lambda^2$$

and we see that there are three equilibria, namely

$$\begin{cases} \varphi_0 = 0 \\ \varphi_1 = \pm\cos^{-1}\left(\dfrac{\omega^2}{\lambda^2}\right), \ \textit{iff}\ \omega^2 < \lambda^2 \\ \varphi_2 = \pi \end{cases}$$

The first of these is stable if $\lambda^2 < \omega^2$, but unstable if the inequality is reversed.

On the other hand, if we let

$$\varphi = \varphi_1 + \chi \, ,$$

we find the equation

$$\ddot{\chi} + \left(\lambda^2 - \omega^2\right)\chi = 0$$

meaning φ_1 is a stable equilibrium point if

$$\lambda^2 > \omega^2 \, ;$$

finally, the equilibrium at π is always unstable.