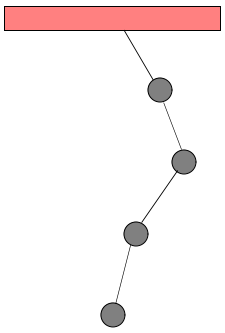
Homework # 10

Due: 11/12/97

1. A uniform chain of length L, made of exceedingly fine links, hangs vertically and executes vibrations in the x–y plane (x is horizontal, y is vertical).

   Find the equation of motion for small horizontal excursions, $\psi(y,t)$, from equilibrium.

   Also determine the frequencies and shapes of the two lowest normal modes.



Solution

We apply the method of lumped parameters: the horizontal displacement of a link, $\psi_n(t)$, obeys Newton's 2nd Law:

$$\mu\Delta y\,\frac{d^2\psi_n}{dt^2} = -\,\frac{T_n}{\Delta y}\left(\psi_n - \psi_{n-1}\right) - \frac{T_{n+1}}{\Delta y}\left(\psi_n - \psi_{n+1}\right).$$

The tension between the n–1'st and n'th links is given by (we number from the top —$n=0$— down)

$$T_n = T_{n+1} + \mu g\Delta y.$$

Converting to a differential equation by letting

$$T_n \to T(n\Delta y)$$

we have

$$T(y+\Delta y) - T(y) = -\mu g\Delta y$$

or

$$\frac{dT}{dy} = -\mu g\,,$$

$$T(y) = \mu\, g(L-y)\,.$$

Performing a similar transition from the difference to a differential equation for $\psi(y,t)$ we find

$$\frac{\partial^2\psi}{\partial t^2} = g(L-y)\,\frac{\partial^2\psi}{\partial y^2} - g\frac{\partial\psi}{\partial y}\,.$$

Clearly, separation of variables is in order: let

$$\psi(y,t) = \varphi(L-y)\,e^{i\omega t}\,;$$

then if we change variables to $u = L-y$,

$$u\frac{d^2\varphi}{du^2} + \frac{d\varphi}{du} + \frac{\omega^2}{g}\varphi = 0\,.$$

Now, although it is not obvious, one can show that the function (J_ν is the ν'th Bessel function)

$$w(x) = x^\alpha J_\nu(\beta x^\gamma)$$

satisfies the differential equation

$$x^2\frac{d^2w}{dx^2} + (1-2\alpha)\,x\,\frac{dw}{dx} + \left(\beta^2\gamma^2x^{2\gamma} + \alpha^2 - \gamma^2\nu^2\right)w = 0$$

Comparing the two we see that

$$\varphi(L-y) = J_0\left(2\omega\sqrt{(L-y)/g}\,\right).$$

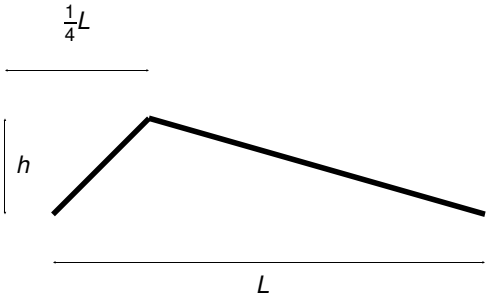
Thus the lowest mode is given by the smallest root, $x = 2.404825\ldots$, of

$$J_0(x) = 0\,,$$

that is,

$$\omega = \frac{2.404825\ldots}{2}\,\sqrt{g/L}\,.$$

2. A uniform stretched string of length L and density μ per unit length, under tension T, is plucked a distance $\frac{1}{4}L$ from one end, so that the initial configuration looks as shown to the right, where $h \ll L$. The string is released at $t=0$.



a) What is the string's subsequent motion?

b) What is the energy of the initial configuration? Calculate the energy in all the normal modes and show that it sums to the initial energy, independent of time.

Solution

The string equation of motion is

$$\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = 0, \quad u^2 = \frac{T}{\mu}$$

Letting $\psi(x,t) \equiv \mathrm{Re}\left[e^{i\omega t}\,\varphi(x)\right]$ as usual, we have

$$\varphi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

so that (the string is initially at rest)

$$\psi(x,t) = \sum_1^\infty A_n \cos\left(\frac{n\pi u}{L}t\right)\sin\left(\frac{n\pi}{L}x\right).$$

To calculate the coefficients A_n we set

$$\psi(x,0) = \sum_1^\infty A_n \sin\left(\frac{n\pi}{L}x\right)$$

and invert using the well-known formula

$$\frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right) = \delta_{mn};$$

this gives

$$A_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi}{L}x\right)\psi(x,0).$$

Proceeding, we have

$$\psi(x,0) = \begin{cases} \frac{4h}{L}x, & 0 < x < \frac{4}{L} \\ \frac{4h}{3L}(L-x), & \frac{4}{L} < x < L \end{cases}$$

so that

$$A_n = \frac{8h}{L^2}\int_0^{L/4} dx\, x \sin\left(\frac{n\pi}{L}x\right) +$$

$$+ (-1)^{n+1}\frac{8h}{3L^2}\int_0^{3L/4} dx\, x \sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{-2h}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - (-1)^n \cos\left(\frac{3n\pi}{4}\right)\right] +$$

$$+ \frac{8h}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{4}\right) - \frac{(-1)^n}{3}\sin\left(\frac{3n\pi}{4}\right)\right]$$

Note that

$$\cos\left(\frac{n\pi}{4}\right) - (-1)^n \cos\left(\frac{3n\pi}{4}\right) \equiv 0 \text{ (prove this).}$$

To solve part b) we note that

$$H = \frac{1}{2}\int_0^L dx\left[\mu\left(\frac{\partial\psi}{\partial t}\right)^2 + T\left(\frac{\partial\psi}{\partial x}\right)^2\right]$$

and that at $t=0$, $\frac{\partial\psi}{\partial t} = 0$, hence

$$E(0) = \frac{T}{2}\int_0^L dx \left(\frac{\partial\psi(x,0)}{\partial x}\right)^2$$

$$= \frac{8}{3}T\frac{h^2}{L}.$$

Now what about the energy at later times? Manifestly,

$$\frac{\mu}{2}\int_0^L dx\,\left(\frac{\partial\psi}{\partial t}\right)^2 = \frac{T}{4L}\sum_1^\infty A_n^2\, n^2\pi^2 \sin^2(\omega_n t)$$

and

$$\frac{T}{2}\int_0^L dx\,\left(\frac{\partial\psi}{\partial x}\right)^2 = \frac{T}{4L}\sum_1^\infty A_n^2\, n^2\pi^2 \cos^2(\omega_n t)$$

so

$$H = \frac{T}{4L}\sum_1^\infty A_n^2\, n^2\pi^2 .$$

This is manifestly time-independent, so we can see that

a) energy is conserved;

b) it is equal to E_0 .

However it is interesting to evaluate the actual sum. We have

$$A_n^2\, n^2\pi^2 = \frac{64h^2}{n^2\pi^2}\begin{cases} \frac{8}{9}, & n = 2k+1 \\ \frac{16}{9}, & n = 2(2k+1) \\ 0, & \text{else} \end{cases}$$

Thus,

$$\sum_{n=1}^\infty A_n^2\, n^2\pi^2 = \frac{64h^2}{\pi^2}\cdot\frac{8}{9}\cdot\frac{3}{2}\cdot\sum_{n=0}^\infty \frac{1}{(2n+1)^2}$$

and thus, since

$$\sum_{n=0}^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

$$\frac{T}{4L}\sum_1^\infty A_n^2\, n^2\pi^2 = \frac{8}{3}\,T\frac{h^2}{L},$$

as before.