Homework # 11

Due: 11/19/97

1. Flutes and recorders can be approximated as long cylinders of uniform cross-section, with one end closed and the other open.

   Obviously the normal component of the velocity field (associated with the displacement of air) in the instrument must vanish at the sides and at the closed end. On the other hand, *pressure* fluctuations must vanish at the open end since that is held at constant (atmospheric) pressure.

   Solve the wave equation subject to the above (mixed) boundary conditions, and thereby determine the lowest (fundamental) mode.

Solution

Let η be the pressure fluctuation; then

$$\frac{1}{u_s^2}\frac{\partial^2 \eta}{\partial t^2} = \nabla^2 \eta$$

and in cylindrical coordinates we have

$$\eta(\vec{x}, t) = A\, J_m(kr)\, \sin(\lambda z)\, e^{im\varphi}\, e^{i\omega t}$$

where we chose the non-singular Bessel function because the pressure remains finite at $r = 0$; we chose $\sin(\lambda z)$ as the solution in the z-direction because the pressure fluctuation must vanish at $z = 0$ (which we choose to be the open end of the recorder); and m and ω are to be determined.

The boundary conditions are mixed—from Newton's Law,

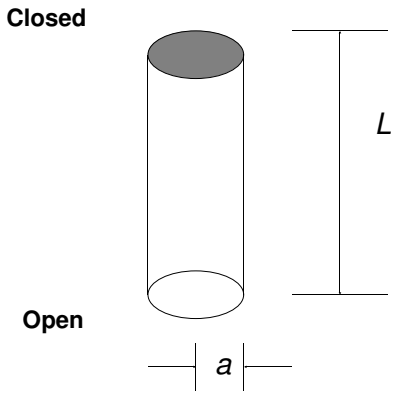
$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p \propto -\nabla \eta$$

and the fact that in harmonic motion,

$$\frac{\partial \vec{v}}{\partial t} = i\omega \vec{v},$$

we see that the vanishing of $\vec{v}_{normal}$ at the walls $\Rightarrow$

$$\hat{z} \cdot \nabla \eta \Big|_{z=L} = 0$$



and

$$\hat{r} \cdot \nabla \eta \Big|_{r=R} = 0.$$

Thus

$$\lambda \cos(\lambda L) = 0$$
$$k\, J'_m(kR) = 0,$$

and

$$\frac{\omega^2}{u_s^2} = k^2 + \lambda^2$$

Manifestly, $\lambda = 0$ is excluded since $\sin(0) \equiv 0$; however, $k = 0$ is allowed for the special case $m = 0$, since $J_0(0) = 1$.

Thus we conclude the allowed frequencies of the fundamental scale are

$$\omega_n = u_s \left(n + \frac{1}{2}\right)\frac{\pi}{L}.$$

   Where would you locate the stops (holes in the side of the instrument) to produce the notes of the 12-tone scale?

Solution

The holes are designed to effectively reduce the length of the tube by forcing the pressure at an open hold to be atmospheric pressure. Thus we must choose the length to be $\sqrt[12]{2} \equiv 1.0594$ times the next shorter length. If the fundamental is chosen

to be 466.16 Hz (A#$_4$), the recorder must be 44.0" long. The holes for B$_4$ *etc.* are spaced at distances (from the upper end) 41.5", 39.2", 37.0", 34.9", 33.0", 31.1", 29.4", 27.7", 26.1", 24.7", 23.3" and 22.0" to get a full octave.

2. Show that if two drumheads have the same shape, but one is uniformly scaled with respect to the other by some factor, that the fundamental mode of the smaller drumhead is higher in pitch than the corresponding mode of the larger, independent of the shape.

   (Hint: use the Rayleigh-Ritz variational principle.)

Solution

The lowest frequency of the larger drumhead is given by the exact formula

$$\Omega^2 = \frac{\iint_A d^2x \nabla\psi_0 \cdot \nabla\psi_0}{\iint_A d^2x \psi_0^2};$$

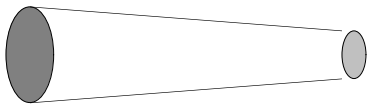
where ψ_0 is the exact eigenfunction. It is bounded above by the same integral, except with the exact eigenfunction replaced by a trial function that vanishes at the boundary. Hence we can choose as our trial function the function

$$\psi = \varphi_0(\vec{x})\,\theta(\vec{x}\in a)$$

where φ_0 is the lowest eigenfunction of the smaller drumhead. Thus,

$$\Omega^2 \le \frac{\iint_A d^2x \nabla\psi \cdot \nabla\psi}{\iint_A d^2x \psi^2} = \frac{\iint_a d^2x \nabla\varphi_0 \cdot \nabla\varphi_0}{\iint_a d^2x \varphi_0^2} \equiv \omega^2$$

3. The Ruritanian zither is a one-stringed instrument whose string tapers uniformly from one



end to the other.

Estimate the fundamental frequency of a zither of length 2 m, whose string is made of silver (specific gravity 10.6) under tension of 50 Nt, and whose end diameters are 1 and 2 mm, respectively.

Solution

We use the Rayleigh-Ritz method, with trial function

$$\varphi(x) = \sin(\tfrac{\pi x}{L}) + \tfrac{\alpha}{2}\sin(\tfrac{2\pi x}{L})$$

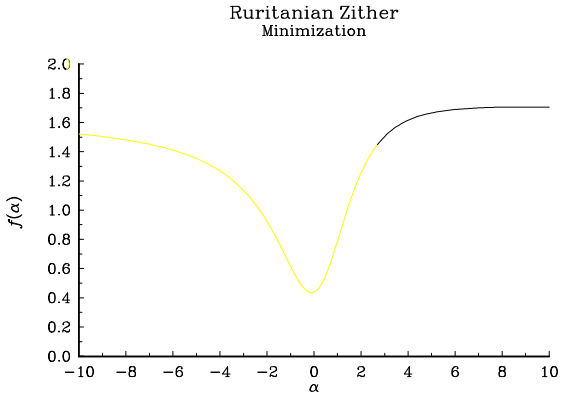
$$\Omega^2 \le \frac{T\int_0^L dx\,[\varphi'(x)]^2}{\int_0^L dx\,\mu(x)[\varphi(x)]^2}$$

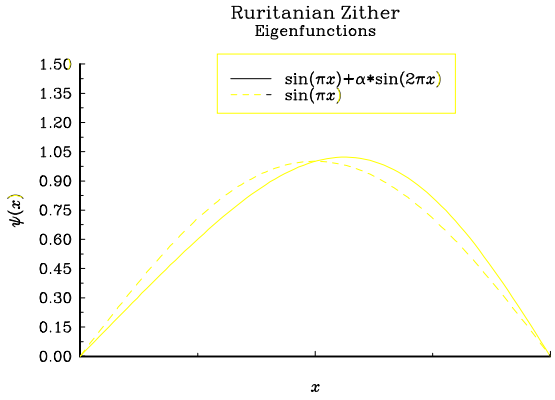
$$= \frac{T\pi^2}{\mu_0 L^2} \times \frac{\int_0^L dx\left[\cos(\tfrac{\pi x}{L}) + \alpha\cos(\tfrac{2\pi x}{L})\right]^2}{\int_0^L dx\left[\sin(\tfrac{\pi x}{L}) + \tfrac{\alpha}{2}\sin(\tfrac{2\pi x}{L})\right]^2 \left(1+\tfrac{x}{L}\right)^2}$$

$$= \frac{T\pi^2}{\mu_0 L^2} \times \frac{\left(1+\alpha^2\right)}{\tfrac{7}{3}\left(1+\tfrac{\alpha^2}{4}\right) + \tfrac{1}{2\pi^2}\left(-1-\tfrac{64}{9}\alpha+\tfrac{7}{16}\alpha^2\right)}$$

to get

The latter function is minimized by α=-0.106188, giving 0.4344, *vs.* 0.4381 at α=0. The corresponding trial function is shown below (the string gets thicker from left to right):





4. A square drumhead of side 1 obviously has (degenerate) normal modes

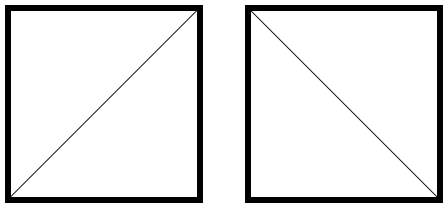
$$\psi_1(x,y) = \sin(\pi x)\sin(2\pi y) - \sin(2\pi x)\sin(\pi y)$$

and

$$\psi_2(x,y) = \sin(\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(\pi y)$$

that are related by a reflection.

Describe qualitatively what happens if a small mass δm is attached at the point $\begin{pmatrix} x=1/4 \\ y=1/4 \end{pmatrix}$.



Estimate the effect of this mass on the vibration frequencies corresponding to these modes.

Hint: First write down the equation of motion of drumhead plus small mass. Treat the mass density as a function of position:

$$\mu(x,y) = \mu_0 + \delta m\, \delta(x-1/4)\, \delta(y-1/4)$$

where $\delta(x)$ is a delta function. The δm term may be regarded as a perturbation to the operator

$$-\nabla^2$$

and determine its value to first order in δm.

Solution

The equation of motion is (after eliminating time dependence by the usual substitution)

$$\mu(x,y)\,\omega^2\,\psi + T\,\nabla^2\psi = 0$$

which can be rewritten, approximately, as

$$\omega^2\,\psi \approx \frac{-T}{\mu_0}\left(1 - \frac{\delta\mu(x,y)}{\mu_0}\right)\nabla^2\psi\,.$$

If we write $\psi = a_1\,\psi_1 + a_2\,\psi_2$ and try to determine the eigenvalue *via* the Rayleigh-Ritz method:

$$\omega^2 \le \frac{-\iint_A dx\,dy\;\psi(x,y)\left(1 - \frac{\delta\mu(x,y)}{\mu_0}\right)\nabla^2\psi(x,y)}{\iint_A dx\,dy\;\psi^2(x,y)}$$

we find

$$\omega^2 \approx \frac{5\pi^2 T}{\mu_0}\left(1 - \frac{4\delta m}{\mu_0}\frac{a_2^2}{a_1^2 + a_2^2}\right);$$

that is, the eigenvalue corresponding to $a_2 = 0$ (pure ψ_1 state) is unchanged, whereas that corre-

sponding to $a_1 = 0$ (pure ψ_2 state) is slightly lowered. The reason is obvious: the pure ψ_1 state has a node where the perturbing mass is placed, hence the mass does not move and cannot affect the frequency. The mirror image mode, however, suffers an increase in effective mass, hence moves more slowly—*i.e.* at lower frequency.

5. The partial pressure of water vapor at 30 °C is about 32 mm of Hg. (See below for P_{vap} *vs.* T_{abs}.)

a) What is the latent heat of the water vapor in one cm^3 of air, at 100% humidity, at 30 °C (in Joules, please, relative to crystallization into ice)?

Solution

We must first determine the total amount of water vapor. The partial pressure is equal to the vapor pressure when the humidity is 100%, hence the fraction of the atmospheric pressure due to water vapor is 32/760=0.0421. That is, the number of mols of water in one cm^3 of saturated air is

$$\frac{32}{760}\times\frac{1}{22.4\times10^3} = 1.88\times10^{-6}\ .$$

From the graph the heat of vaporization is

$$573\ \text{cal/gm} \times 4.185\ \text{J/cal} \times 18\ \text{gm/mol} \times$$

$$\times\ 1.88\times10^{-6}\ \text{mol/cm}^3 = 0.0811\ \text{J/cm}^3$$

The heat of fusion of water is 79.71 cal/gm, giving a total latent heat of the water in saturated air at 30 °C of 0.0924 J/cm^3.

b) Estimate the heat energy available in a cylindrical air mass of radius 100 km and height 1 km.

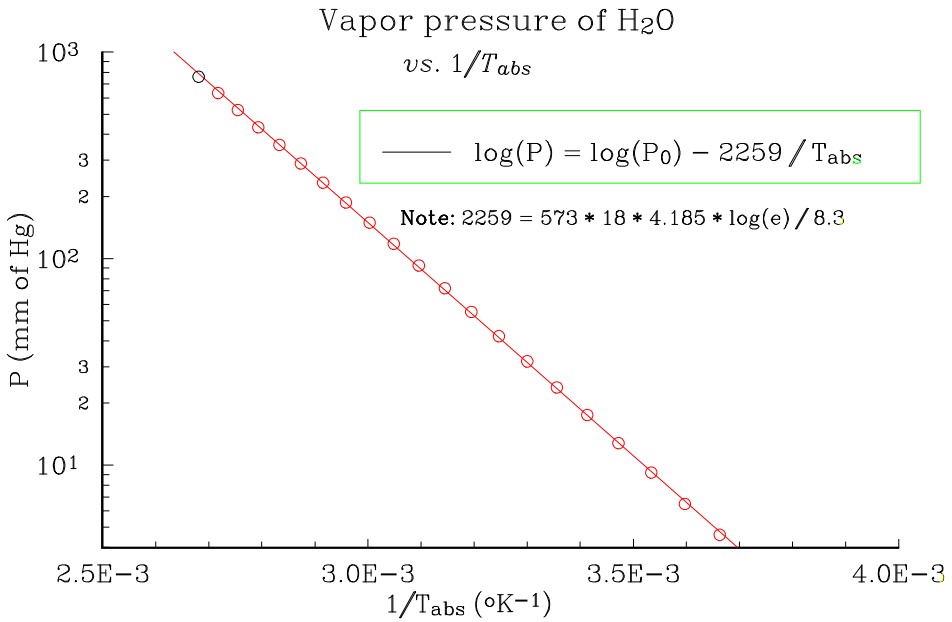
Solution

Neglecting the variation of pressure and temperature altitude we find 3×10^{18} J, in round numbers.

c) What is the maximum fraction of this energy that can be converted to kinetic energy?

Solution

The Carnot efficiency is just the temperature change divided by the (larger) of the absolute temperatures, i.e.



$$\eta = \frac{30}{303} \approx 0.1$$

meaning at best 3×10^{17} J are available.

d) A 1 MT (megaton) (thermo)nuclear explosive releases 1.2×10^{9} kw–hr of energy. Express the kinetic energy of the storm system just described in MT. Estimate the average air speed.

Solution

The above storm system is about the equivalent of 70 1 MT bombs, or 3500 Hiroshima-size nuclear bombs.

The average air speed is obtained by assuming the available energy is converted into kinetic energy by convection, that is, the kinetic energy will be, allowing for Carnot efficiency 0.00924 J/cm^3. But this volume of air has mass 1.3×10^{-6} kg, giving a speed of 119 m/sec, or 267 mi/hr. Manifestly, we now understand the origin of tornados!