Homework #12

1. What is the velocity of propagation of small-amplitude surface waves on a body of inviscid fluid that has depth h, but is infinite in either of the horizontal directions?

Solution

The solution proceeds as for an infinite-depth body of fluid. Since $\widetilde{\varphi}$ satisfies the Laplace equation, and since we want a wavelike solution in the horizontal direction, let

$$\widetilde{\varphi} = f(z)\, e^{i(kx - \omega t)} ,$$

leading to

$$-k^2 f(z) + \frac{d^2 f}{dz^2} = 0 .$$

Since we do not go to $z \to -\infty$, we take

$$f(z) = A \sinh kz + B \cosh kz .$$

The normal component of velocity must vanish at $z = -h$, hence

$$kA \cosh kh - kB \sinh kh = 0 ,$$

or

$$f(z) = A\left(\sinh kz + \coth kh \cosh kz\right).$$

The boundary condition at the free surface is again

$$\left[\frac{1}{g}\frac{\partial^2 \widetilde{\varphi}}{\partial t^2} + \frac{\partial \widetilde{\varphi}}{\partial z}\right]_{z=0} = 0 ,$$

or

$$\omega^2 = \frac{kg}{\coth kh} \equiv kg \tanh kh .$$

The corresponding group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\left[\left(\frac{g \tanh kh}{k}\right)^{1/2} + h\left(\frac{gk}{\sinh kh \cosh kh}\right)^{1/2}\right]$$

2. A spherical hole of radius a is suddenly formed in an infinite body of incompressible inviscid fluid (not in a graviational field). Assume the pressure at $r = \infty$ is $p(\infty) =$ constant. How long does it take for the hole to disappear?

Solution

We apply conservation of mass,

$$\nabla \cdot \vec{v} = 0,$$

and Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\nabla\right)\vec{v} = -\frac{1}{\rho}\nabla p\ .$$

The velocity field is, by symmetry, radial only, *i.e.* $\vec{v} = v(r,t)\,\hat{r}$. Thus mass conservation yields

$$r^2 v = f(t)$$

and Euler's equation gives

$$\frac{\dot{f}}{r^2} - 2\frac{f^2}{r^5} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0\ .$$

Integrating once with respect to r we find

$$-\frac{\dot{f}}{r} + \frac{f^2}{2r^4} + \frac{p}{\rho} = \frac{p_\infty}{\rho}$$

Now we evaluate at $r = R(t)$, the instantaneous radius of the hole. We note that at this radius, $v = \frac{dR}{dt}$ and $p = 0$, leading to

$$\frac{3v^2}{2} + R\frac{dv}{dt} = \frac{p_\infty}{\rho}$$

or since

$$\frac{dv}{dt} \equiv \frac{dv}{dR}\frac{dR}{dt} = \frac{1}{2}\frac{d(v^2)}{dR},$$

$$\frac{d(v^2)}{dR} + \frac{3}{R}v^2 = \frac{2p_\infty}{\rho R}$$

which can be solved easily to obtain (note that when $R=a$, $v=0$)

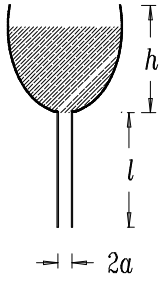
$$v^2 = \frac{2}{3}\frac{p_\infty}{\rho}\left(1 - \frac{a^3}{R^3}\right),$$

and *via* one further integration,

$$t = \left(\frac{3\rho a^2}{2p_\infty}\right)^{1/2}\int_0^1 dx\, x^{3/2}\left(1-x^3\right)^{-1/2} = \left(\frac{\rho a^2}{6p_\infty}\right)^{1/2}\int_0^1 dy\, y^{-1/6}\left(1-y\right)^{-1/2}$$

$$= \left(\frac{\rho a^2}{6p_\infty}\right)^{1/2} B(5/6,\ 1/2)\ .$$

3. Consider a cylindrically symmetric object (wineglass) of arbitrary shape, $r = R(z)$ and height h, connected as shown to a concentric tube of constant radius a and length l. The glass is initially filled to the top with a fluid of viscosity η and density ρ. At this moment, whatever is keeping it from flowing through the tube is removed. How long does it take for the glass to empty? (Assume that $R(0) = 0$.)



Solution

We assume the bottleneck (as it were) is the tube, *i.e.* that we may neglect viscous effects in the glass proper. We shall apply Poiseuille's Law in the form

$$\frac{dM}{dt} = \frac{\pi\rho a^4}{8\eta} \cdot \frac{p - p_a}{l},$$

first to the liquid initially in the glass, then to the liquid in the tube.

The pressure at the top of the tube is $\rho g z + p_a$, and with the instantaneous mass of fluid in the glass given by

$$M(z) = \rho\,\pi \int_0^z d\zeta\, R^2(\zeta),$$

we see that

$$\frac{dM}{dt} = \rho\,\pi\, R^2(z)\,\frac{dz}{dt} = \frac{\pi\rho a^4}{8\eta l}\,\rho g z$$

giving

$$t_{glass} = \frac{8\eta\, l}{\rho g a^4} \int_0^h \frac{dz}{z}\, R^2(z).$$

Manifestly this integral can only be finite if $R(0) = 0$.

Now, what about the time for the tube to drain? Applying Poiseuille's Law,

$$\frac{dM}{dt} = \rho\,\pi\, a^2\,\frac{dz}{dt} = \frac{\pi\rho a^4}{8\eta z}\,\rho g z,$$

leading to

$$t_{tube} = \frac{8\eta\, l}{\rho g a^2}.$$

The total time is the sum of these.

4. Consider the nonstatic case of the one-dimensional atomic model (masses connected by springs) above. Write the equation of motion for the n'th atom's displacement from equilibrium, $\xi_n(t)$.

   Solve the equation of motion and determine the dispersion relation for travelling waves. What is the maximum frequency the system can sustain?

Solution

If the masses are identical, $m_k = \delta m$ and the spring constants as well, then the equation of motion for each displacement is

$$\delta m\, \ddot{\xi}_n = k\left(\xi_{n+1} - 2\xi_n + \xi_{n-1}\right) \equiv k\,\Delta^2\,\xi_n .$$

With the usual substitution,

$$\xi_n(t) = e^{i\omega t}\,\bar{\xi}_n ,$$

we obtain

$$\Delta^2\,\bar{\xi}_n + \frac{\omega^2}{\Omega^2}\,\bar{\xi}_n = 0 .$$

By analogy with ordinary linear differential equations, linear difference equations have the solutions

$$\bar{\xi}_n = A\,a^n$$

where $\Omega^2 = \dfrac{k}{\delta m}$ and

$$a - 2\left(1 - \frac{\omega^2}{2\Omega^2}\right) + \frac{1}{a} = 0 .$$

The roots are obviously complex and of modulus unity, hence let

$$a = e^{-i\varphi}$$

where

$$\cos\varphi = 1 - \frac{\omega^2}{2\Omega^2} .$$

This imposes a frequency cutoff of $\omega_{\max} = 2\Omega$, in order that the modulus of the cosine be smaller than unity. Travelling waves are described by

$$\xi_n(t) = A\,e^{i(\omega t - n\varphi)}$$

where $n = \dfrac{x}{\delta x}$. The dispersion relation (connection between wave number and frequency) for such waves is then

$$k = \frac{2}{\delta x}\sin^{-1}\left(\frac{\omega}{2\Omega}\right).$$

5. Prove that a transformation of the basis,

$$\hat{e'} = A\,\hat{e}$$

that diagonalizes a real-symmetric matrix M, is orthogonal, *i.e.*

$$\tilde{A}\,A = I$$

where $\tilde{A}$ is the transpose of A.

Solution

This is almost trivial. We know that a real-symmetric $N\times N$ matrix has N real eigenvalues and N corresponding eigenvectors. (We presume degeneracies have been removed by appropriate symmetry-breaking small perturbations.) Thus,

$$M\,u_k = m_k\,u_k\ .$$

If we suppose the eigenvectors normalized as well as orthogonal,

$$(u_k, u_l) = \delta_{kl}\ ,$$

we can immediately construct the diagonalizing matrix A:

$$A = \left(\begin{pmatrix} \\ u_1 \\ \\ \end{pmatrix} \begin{pmatrix} \\ u_2 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ u_N \\ \\ \end{pmatrix} \right)$$

—that is, A is a matrix each of whose columns is a normalized eigenvector. Obviously, A is orthogonal: $\tilde{A}\,A = I$; it is also easy to see that

$$M\,A = \left(m_1 \begin{pmatrix} \\ u_1 \\ \\ \end{pmatrix} \; m_2 \begin{pmatrix} \\ u_2 \\ \\ \end{pmatrix} \cdots \; m_N \begin{pmatrix} \\ u_N \\ \\ \end{pmatrix} \right)$$

so that

$$\tilde{A}\,M\,A = \begin{pmatrix} m_1 & 0 & \cdots & \cdots \\ 0 & m_2 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & m_N \end{pmatrix}.$$

6. A board (thin beam) of thickness t and width w is clamped at one end (at the top of a wall) and extends a length l beyond the wall. How much does the free end deflect under its own weight?

   If the wood is white pine, 0.75 inches thick and 12 feet long (specific gravity 0.46, Young's modulus 8×10^{10} dyne/cm^2) what is the actual deflection?

Solution

As discussed in class (and as can be discerned from p. 137 of the notes), the deflection energy of a thin, uniform beam is given by

$$E = \frac{Y I}{2}\int_0^L \left(\frac{d^2\xi}{dz^2}\right)^2 dz$$

where I is the "areal moment of inertia",

$$I = \int x^2\, dxdy\ .$$

For the uniform beam, $I = \frac{wt^3}{12}$; the total energy of the beam, incuding the effect of gravity, is

$$E_{tot} = \int_0^L \left[\frac{Y I}{2}\left(\frac{d^2\xi}{dz^2}\right)^2 + \rho\, wt\, g\xi(z)\right] dz\ .$$

Variation with respect to $\xi(z)$ gives

$$\delta E_{tot} = \int_0^L \left[g\rho wt + YI\frac{d^4\xi}{dz^4}\right]\delta\xi(z)\, dz +$$

$$+ YI\left(\frac{d^3\xi}{dz^3}\frac{d}{dz}\delta\xi - \frac{d^2\xi}{dz^2}\delta\xi\right)\Bigg|_0^L = 0\ .$$

Since the variation is arbitrary, we find first, that the integrand must vanish almost everywhere, or

$$g\rho wt + YI\frac{d^4\xi}{dz^4} = 0\ .$$

Because the end at $z=0$ is clamped, both the variation $\delta\xi(z)$ and its first derivative vanish there. Thus there is no end-point contribution from that end. However, since there is no constraint on either the variation or its derivative at the end $z=L$, in order that the total variation vanish, both the second and third derivatives of ξ must vanish there. The four boundary conditions needed to specify completely the solution are

$$\xi(0) = \frac{d}{dz}\xi(0) = 0$$

$$\left.\frac{d^2\xi}{dz^2}\right|_{z=L} = \left.\frac{d^3\xi}{dz^3}\right|_{z=L} = 0\,.$$

The solution of the 4th order differential equation is obviously

$$\xi(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 - \frac{g\rho}{2Yt^2} z^4$$

where we use the two conditions at $z=0$ to set

$a_0 = a_1 = 0$;

and the two conditions at $z=L$ to set

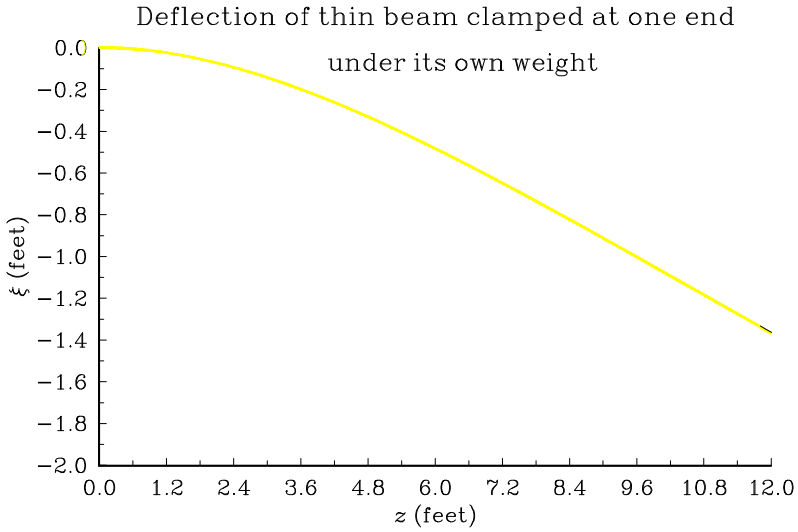
$$a_2 = \frac{-3g\rho L^2}{Yt^2}$$

$$a_3 = \frac{2g\rho L}{Yt^2}\,.$$

Thus

$$\xi(z) = \frac{-g\rho}{Yt^2} z^2 \left(3L^2 - 2Lz + \frac{z^2}{2}\right)$$

and the deflection at the unsupported end is

$$\xi(L) = \frac{-3g\rho L^4}{2Yt^2} = \frac{-1.5 \times 980 \times 0.46 \times (144)^4 \times (2.54)^4}{8\times 10^{10} \times (0.75)^2 \times (2.54)^2} = -41.7 \text{ cm}$$

which is about –16.4". The curve looks like this: