# Homework #13

1. A beam (not necessarily thin) is clamped at both ends. It extends horizontally in the $z$–direction, and gravity acts to deflect it downwards ($-x$-direction).

   a) Find the deflection $\xi(z)$ under its own weight.

   b) A weight $W$ is hung from the center. What is $\xi(z)$ now?

**Solution**

Since it is clamped at both ends, the deflection satisfies

$$\xi(0) = \xi(L) = 0\,,$$

$$\xi'(0) = \xi'(L) = 0\,.$$

For part a) we vary

$$E_{tot} = \int_0^L dz \left[ \frac{YI}{1-\sigma^2} \left( \frac{d^2\xi}{dz^4} \right)^2 + \rho A g \xi(z) \right]$$

giving

$$\frac{d^4\xi}{dz^4} + \lambda = 0\,, \quad \lambda = \frac{1-\sigma^2}{YI}\rho A g\,.$$

Manifestly,

$$\xi(z) = a_2 z^2 + a_3 z^3 - \frac{\lambda}{24} z^4$$

hence

$$\frac{\xi(L)}{L^2} = a_2 + a_3 L - \frac{\lambda}{24} L^2 = 0\,,$$

$$\frac{\xi'(L)}{L} = 2a_2 + 3a_3 L - \frac{\lambda}{6} L^2 = 0\,.$$

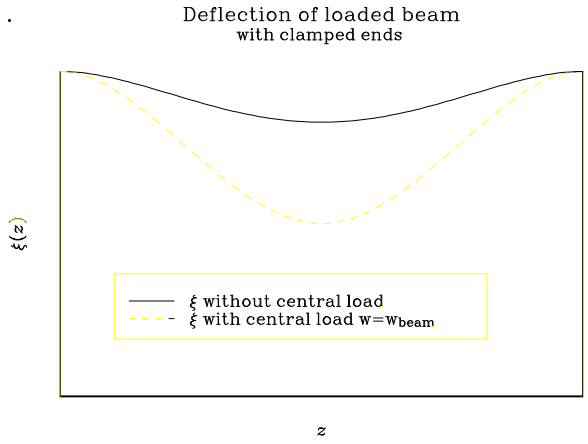
Solving,

$$a_3 = \frac{\lambda L}{12}\,,$$

$$a_2 = \frac{-\lambda L^2}{24}\,,$$

so

$$\xi(z) = \frac{-\lambda}{24} z^2 \left( z - L \right)^2\,.$$



b) If a weight $w$ is hung from the center, the boundary conditions remain the same, but the equation becomes

$$\frac{d^4\xi}{dz^4} + \lambda + \gamma\,\delta(z-\tfrac{L}{2}) = 0\,,\quad \lambda = \frac{1-\sigma^2}{YI}\rho Ag\,,\quad \gamma = \frac{1-\sigma^2}{YI}w\,.$$

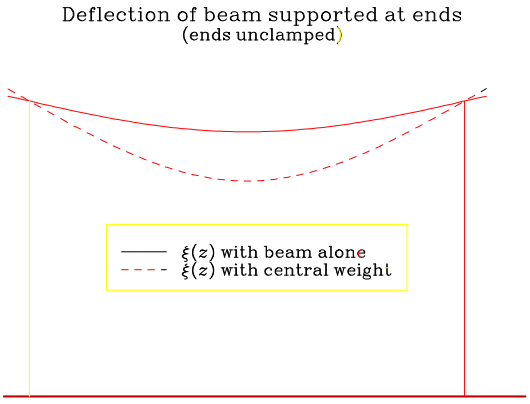
Consequently, the solution is

$$\xi(z) = \frac{-\lambda}{24}z^2\left(z-L\right)^2 - \frac{\gamma}{6}\left(z-\tfrac{1}{2}L\right)^3\theta\left(z-\tfrac{1}{2}L\right) + \frac{\gamma z^2}{12}\left(z-\tfrac{3}{4}L\right)$$

(we can see this by integrating the δ-function term from 0 to $z$ four times). Note that

$$\int_0^z dz'\;\delta\left(z'-\tfrac{1}{2}L\right) = \theta\left(z-\tfrac{1}{2}L\right) = \begin{cases} 0\,, & z < \frac{1}{2}L \\ 1\,, & z \geq \frac{1}{2}L \end{cases}\,.$$

2. A horizontal beam is supported at both ends, but the ends are not clamped. If a weight $w$ is hung from the middle, what is $\xi(z)$?




**Solution**

Because the ends are not clamped, we can only require the variation to vanish at the end points and not its first derivative. Hence $\xi(z)$ and its second derivative must vanish at the ends. The equation of static equilibrium is as above, giving

$$\xi(z) = \frac{-\lambda z}{24}\left(z^3 - 2Lz^2 + L^3\right) -$$

$$\frac{-\gamma}{6}\left(z-\tfrac{1}{2}L\right)^3\theta\left(z-\tfrac{1}{2}L\right) + \frac{\gamma z}{12}\left(z^2 - \tfrac{3}{4}L^2\right)$$

3. A vertical (cylindrical) pillar supports a weight $W$ that is distributed uniformly on its upper surface. Write the equations for compressive deflection as well as any deflection in the horizontal direction.

**Solution**

As we saw on p. 141 of The Notes, the compressive deflection is

$$\frac{\partial u_z}{\partial z} \approx -\frac{1}{2}\left(\frac{d\xi}{dz}\right)^2 - \frac{W}{AY}$$

The transverse deflection satisfies the equation

$$\frac{d^4\xi}{dz^4} + \frac{W}{YI}\frac{d^2\xi}{dz^2} = 0 .$$

a) What is the maximum weight that can be sustained before the pillar becomes unstable?

**Solution**

As we saw in The Notes, the maximum weight is determined by the first non-zero eigenvalue,

$$\frac{W}{YI} = \frac{\pi^2}{L^2};$$

for a solid cylinder, $I = \int_0^{2\pi} d\varphi \int_0^R dr\, r\, x^2 \equiv \frac{\pi}{4} R^4$, hence

$$W_{\max} = \frac{\pi^3 Y R^4}{4L^2} .$$

b) Suppose the pillar is a hollow cylinder with thin walls ($t = r_2 - r_1 << r_2$). What is the maximum weight now?

**Solution**

For the hollow cylinder, $I \approx \pi R^3 t$, so the maximum load is reduced by a factor $1 - \left(1 - \frac{t}{R}\right)^4 \approx \frac{4t}{R}$. The ratio of compressive strength to weight then is *increased* by the factor $1 + \left(1 - \frac{t}{R}\right)^2 \approx 2$.

4. A thin cylindrical shaft of radius $R$ and length $L$ rotates at angular frequency $\omega$.

a) What is the maximum value of $\omega$ before the shaft becomes unstable with respect to bowing (that is, deflection perpendicular to the shaft)?

**Solution**

We may suppose that any transverse displacement $\xi(z)$ vanishes at the ends of the shaft (bearings) as does the first derivative. The easiest way to set up the problem is to write the Lagrangian in terms of the kinetic energy of a mass $\rho\, d^3x$ located a distance $x$ from the centerline along the body-fixed $x$-axis. Then we have

$$L = \int_0^L dz \left[ \int_0^{2\pi} d\varphi \int_0^{R(\varphi, z)} dr r \frac{1}{2} \rho\, r^2\, \omega^2 - \frac{YI}{2}\left(\frac{d^2\xi}{dz^2}\right)^2 \right]$$

where

$$R(\varphi\,,z) = R\ +\ \xi(z)\cos\varphi$$

and

$$\omega = \dot{\varphi} = \text{constant}\,.$$

Thus we must vary the (time-independent) expression

$$L = \int_0^L dz \left[ \frac{3}{4}\pi\rho\, R^2\,\omega^2\,\xi^2\ -\ \frac{YI}{2}\left(\frac{d^2\xi}{dz^2}\right)^2 \right] + O\left(\xi^4\right)$$

obtaining

$$\frac{3}{4}\pi\rho\, R^2\,\omega^2\,\xi\ -\ YI\,\frac{d^4\xi}{dz^4}\ =\ 0$$

with boundary conditions

$$\xi(0) = \xi(L) = 0$$

$$\xi'(0) = \xi'(L) = 0\,.$$

Letting

$$k^4\ =\ \frac{3\pi\rho\omega^2\,R^2}{4YI}$$

we find

$$\left(\frac{d^4}{dz^4}\ -\ k^4\right)\xi \equiv \left(\frac{d^2}{dz^2}\ -\ k^2\right)\left(\frac{d^2}{dz^2}\ +\ k^2\right)\xi\ =\ 0$$

or

$$\xi(z)\ =\ A\sin kz\ +\ B\cos kz\ +\ C\sinh kz\ +\ D\cosh kz\,.$$

The boundary conditions at $z=0$ give

$$A\ +\ C\ =\ 0$$
$$B\ +\ D\ =\ 0$$

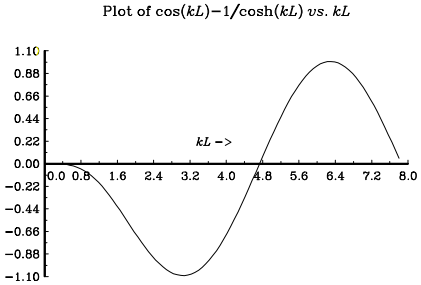
and those at $z=L$ give

$$\begin{pmatrix} \sin\theta - \sinh\theta & \cos\theta - \cosh\theta \\ \cos\theta - \cosh\theta & -\sin\theta - \sinh\theta \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

In order to have a non-trivial solution for the deformation, we must have either $k=0$ or one of the (non-zero) roots of

$$\cos kL\cdot\cosh kL\ =\ 1,$$

of which the smallest is $k_1 L \approx 4.725$, as we see from the plot to the right. When the angular velocity reaches the critical value



$$\omega_{crit} = \frac{\theta_1^2}{RL^2}\left(\frac{4YI}{3\pi\rho}\right)^{1/2} = \frac{\theta_1^2 R}{L^2}\left(\frac{Y}{3\rho}\right)^{1/2}$$

the shaft spontaneously deforms. Note that the critical angular velocity is smaller for a skinny shaft than for a thick one, and for a more flexible shaft than for one made of a stiffer material.

The basic physics here is that when the centrifugal force for some displacement exceeds the elastic restoring force, the shaft can spontaneously deform. This is why a denser material will have a lower critical angular velocity than a lighter one.

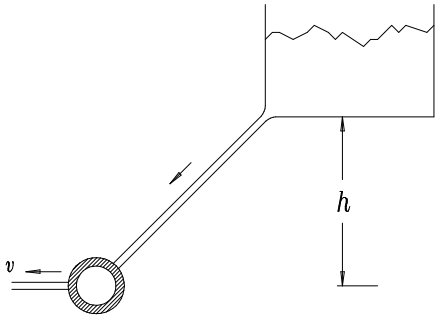
b) What would be the answer to a) for a thin-walled cylindrical shaft?

**Solution**

Following the same method as above but integrating from an inner radius $R$-$t$ to $R$, we discover the critical angular velocity is multiplied by the factor that appeared in the strength-to-weight ratio, but to the 1/2 power:

$$\omega_{crit}(\text{hollow}) = \omega_{crit}(\text{solid})\left(\frac{1-(1-t/R)^4}{1-(1-t/R)^2}\right)^{1/2} \approx \omega_{crit}(\text{solid})\sqrt{2}\ .$$

5. Water falls from a reservoir located a height $h$ above a turbine and exits the turbine at speed $v$. What speed corresponds to producing the maximum power from the turbine?




**Solution**

Water falling a distance h gains a speed $u = \sqrt{2gh}$ ; hence if an amount $dm$ falls through the turbine, emerging with speed $v$, the energy deposited in the turbine is

$$dE = \frac{1}{2}dm\left(u^2 - v^2\right) = \frac{1}{2}dm\left(2gh - v^2\right).$$

The power is then

$$P = \frac{dE}{dt} = \frac{1}{2}\dot{m}\left(2gh - v^2\right) = \frac{1}{2}\rho A\, v\left(2gh - v^2\right),$$

where $A$ is the area of the pipe exiting the turbine. The function $v\left(2gh - v^2\right)$ is maximized at $v^2 = \frac{2gh}{3}$, meaning only 1/3 of the available potential energy can be utilized!