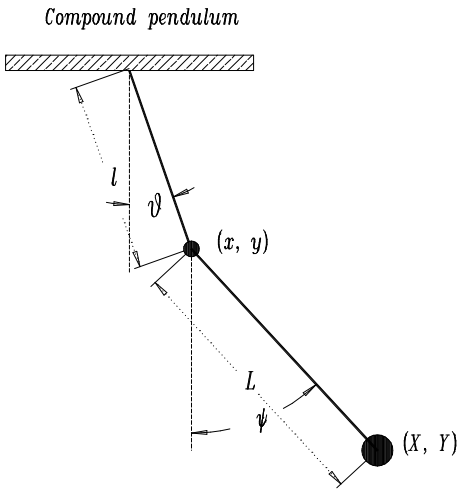
Do all 3 problems. Due: Friday, 30 October 1997

1. A double pendulum is confined to swing in the x-y plane, as shown to the right.



   a) Write down the Lagrangian and find the Euler-Lagrange equations describing its motions, expressed in terms of appropriate generalized coordinates.

   b) Assuming small displacements from equilibrium, linearize the equations of motion.

   c) Find the possible frequencies for small oscillations.

Solution:

$$x = l\sin\theta$$
$$y = -l\cos\theta$$
$$X = x + L\sin\psi$$
$$Y = y - L\cos\psi$$

hence the Lagrangian is

$$\boldsymbol{L} = \frac{1}{2}(m+M)\,l^2\,\dot\theta^2 + \frac{1}{2}M\,L^2\,\dot\psi^2 + MlL\cos(\theta-\psi)\,\dot\theta\dot\psi + (m+M)\,g\,l\cos\theta + M\,g\,L\cos\psi\,.$$

The Euler-Lagrange equations are

$$(m+M)\,l^2\,\ddot\theta + MlL\frac{d}{dt}\Big[\cos(\theta-\psi)\,\dot\psi\Big] = -(m+M)\,g\,l\sin\theta - MlL\sin(\theta-\psi)\,\dot\theta\dot\psi$$

and

$$M\,L^2\,\ddot\psi + MlL\,\frac{d}{dt}\Big[\cos(\theta-\psi)\,\dot\theta\Big] = -MgL\sin\psi - MlL\sin(\theta-\psi)\,\dot\theta\dot\psi$$

Assuming the angles θ and ψ (as well as the corresponding velocities) are small, we then have (to first order in angles)

$$(m+M)\,l^2\,\ddot\theta + MlL\,\ddot\psi = -(m+M)\,g\,l\,\theta$$
$$M\,L^2\,\ddot\psi + MlL\,\ddot\theta = -MgL\,\psi\,.$$

Letting $\omega^2 = g/l$ and $\Omega^2 = g/L$, and substituting

$$\begin{pmatrix}\theta(t)\\ \psi(t)\end{pmatrix} = \begin{pmatrix}\theta_0\\ \psi_0\end{pmatrix}e^{i\lambda t},$$

we find

$$\det\begin{pmatrix} \omega^2-\lambda^2 & -\lambda^2\dfrac{ML}{l(m+M)} \\ -\lambda^2\dfrac{l}{L} & \Omega^2-\lambda^2 \end{pmatrix} = 0$$

The secular equation is thus

$$\left(\omega^2-\lambda^2\right)\left(\Omega^2-\lambda^2\right) - \lambda^4\,\frac{M}{m+M} = 0\,;$$

the roots of this quadratic equation are both real and positive:

$$\lambda^2 = \frac{1}{2}\left(1+\frac{M}{m}\right)\left(\omega^2+\Omega^2\right)\left[1 \pm \left(1 - \frac{4\omega^2\Omega^2}{\left[\omega^2+\Omega^2\right]^2\left[1+M/m\right]^2}\right)^{1/2}\right],$$

hence the motion is sinusoidal.

2. A particle moves in a circular orbit about an attractive force center described by the potential function $V(r)$.

   a) Write the Euler-Lagrange equations of motion.

   b) Find the two constants of the motion.

   c) Consider small perturbations, $r(t) = r_0 + \rho(t)$, $\varphi(t) = \omega t + \psi(t)$ where ρ and ψ are small. If $V(r) = \dfrac{-const.}{r^n}$, for what range of n is the motion stable?

Solution:

The Lagrangian is (after eliminating the θ-dependence in the usual manner)

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)\,;$$

hence the Euler-Lagrange equations are

$$mr^2\dot{\varphi} = J = \text{constant}$$

and

$$m\ddot{r} = -\frac{\partial V}{\partial r} + \frac{J^2}{mr^3}\,.$$

This can be integrated immediately to give the energy:

$$\frac{1}{2}m\dot{r}^2 + V(r) + \frac{J^2}{2mr^2} = E = \text{constant}.$$

From the first E-L equation we find

Pledge: On my honor I have neither given nor received assistance on this examination.

$$m(r_0+\rho)^2(\omega+\dot{\psi}) = mr_0^2\,\omega,$$

leading to the connection

$$2\rho\,\omega + r_0\,\dot{\varphi} = 0.$$

The radial equation yields (note that

$$\frac{J^2}{mr_0^3} = \left.\frac{dV}{dr}\right|_{r_0}$$

because the total force must vanish in a circular orbit)

$$m\ddot{\rho} + \rho\left[V''(r_0) + \frac{3}{r_0}V'(r_0)\right] = 0.$$

Therefore the condition for stability is that the quantity in square brackets be positive. For a radial potential of the form

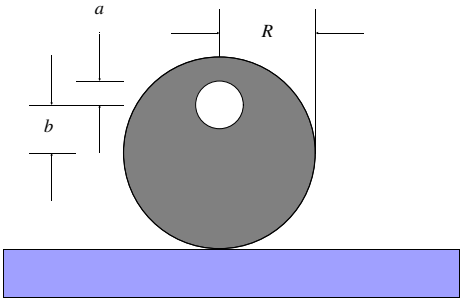
$$V(r) = \frac{-k}{r^n}$$

this quantity becomes

$$\frac{-k}{r_0^{n+2}}\Big(n(n+1) - 3n\Big)$$

so that the motion is stable *iff* $0 < n < 2$.

3. A uniform cylinder of radius R rolls without slipping on a plane, as shown. The cylinder has a round hole of radius $a < R$ drilled through it parallel to the axis. The center of the hole is a distance b from the axis.



a) find the equilibrium position(s) of the (drilled) cylinder and determine which, if any, are stable;

b) find the frequency of small oscillations about the position of stable equilibrium.

***Hint**: treat the hole as a cylinder of negative mass!*

Solution:

The moment of inertia of a cylinder of uniform density about its axis is $\frac{1}{2}mr^2$.

The kinetic energy of each cylinder is composed of its rotational and translational parts. Thus, the big cylinder's KE is

$$T_R = \frac{1}{2} I_R \dot{\theta}^2 + \frac{1}{2} M \dot{X}^2$$

and that of the small cylinder is

$$T_a = \frac{1}{2} I_a \dot{\theta}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right),$$

where

$$I_R = \frac{1}{2} M R^2, \; I_a = \frac{1}{2} m a^2, \text{ and } m = -M \frac{a^2}{R^2}.$$

Manifestly,

$$\dot{X} = R\dot{\theta},$$

$$\dot{x} = \dot{X} + b\dot{\theta} \cos\theta,$$

and

$$\dot{y} = -b\dot{\theta} \sin\theta$$

so the Lagrangian becomes

$$L = \frac{1}{2}\dot{\theta}^2 \left[\frac{3}{2} M R^2 + \frac{1}{2} m a^2 + m\left(R^2 + 2bR\cos\theta + b^2 \right) \right] - mgb\cos\theta$$

Symmetry tells us there are only two candidates for equilibria, namely $\theta=0$ and $\theta=\pi$. We can expand the Lagrangian about these two points to get

$$L_0 = \frac{1}{2} I_0 \dot{\theta}^2 + \frac{1}{2} mgb\theta^2$$

$$L_\pi = \frac{1}{2} I_\pi \dot{\theta}^2 - \frac{1}{2} mgb\theta^2$$

where if

$$I_0 = M\left[\frac{3}{2} R^2 - \frac{a^2}{R^2}\left(\frac{a^2}{2} + (R+b)^2 \right) \right] > 0,$$

then surely

$$I_\pi = M\left[\frac{3}{2} R^2 - \frac{a^2}{R^2}\left(\frac{a^2}{2} + (R-b)^2 \right) \right] > 0$$

also. But it is easy to see that because $a + b < R$, the first inequality is satisfied, hence, since $m < 0$, top dead center is the stable point and bottom dead center is the unstable point.

The frequency of small oscillations is therefore

$$\omega^2 = \frac{a^2}{R^2} gb \left[\frac{3}{2} R^2 - \frac{a^2}{R^2}\left(\frac{a^2}{2} + (R+b)^2 \right) \right]^{-1}.$$