HW #2—Solutions

1. The space-time points A, B have a timelike separation. Show that the integral

$$\int_A^B d\tau \equiv \int_A^B \sqrt{dx^\mu\, dx_\mu}$$

is maximum when the world-line $x^\mu(\tau)$ between the points is a straight line.

Solution:

We parameterize space-time paths linking A with B by $x^\mu(\lambda)$, so that

$$T = \int_A^B d\tau = \int_A^B d\lambda \left(\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \eta_{\mu\nu} \right)^{1/2} \equiv \int_A^B d\lambda \sqrt{\dot{x} \cdot \dot{x}},$$

and we want to maximize T. This is a problem in calculus of variations. If we define

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

and suppose $x^\mu(\lambda)$ is the maximizing path, then if we add a small increment $\xi^\mu(\lambda)$ to $x^\mu(\lambda)$, then the change in T, is, through second order in ξ,

$$\delta T = \int_A^B \frac{d\lambda}{\frac{d\tau}{d\lambda}} \left[\dot{x} \cdot \dot{\xi} + \frac{\frac{1}{2}\left(\dot{x} \cdot \dot{x}\ \ \dot{\xi} \cdot \dot{\xi} - (\dot{x} \cdot \dot{\xi})^2 \right)}{\left(\frac{d\tau}{d\lambda} \right)^2} \right] + \ldots$$

where

$$\frac{d\tau}{d\lambda} = \sqrt{\dot{x} \cdot \dot{x}},$$

and the dot product is shorthand for contraction:

$$\dot{x} \cdot \dot{\xi} \equiv \frac{dx^\mu}{d\lambda} \frac{d\xi_\mu}{d\lambda}.$$

The increment $\xi^\mu(\lambda)$ vanishes at the points A and B.

If T is to be an extremum, then the first-order term in $\xi^\mu(\lambda)$ must vanish. That is,

$$\delta T^{(1)} = \int_A^B \frac{d\lambda}{\sqrt{\dot{x} \cdot \dot{x}}}\ \dot{x} \cdot \dot{\xi} = 0,$$

or upon integrating by parts,

$$\delta T^{(1)} = \left. \frac{\xi \cdot \dot{x}}{\sqrt{\dot{x} \cdot \dot{x}}} \right|_A^B - \int_A^B d\lambda\, \xi \cdot \frac{d}{d\lambda} \left(\frac{\dot{x}}{\sqrt{\dot{x} \cdot \dot{x}}} \right) = 0$$

The first term vanishes because ξ vanishes at the endpoints. For the second term to vanish, because x is arbitrary, it must be true that

$$\frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}\cdot\dot{x}}}\right) = 0\,.$$

It is now appropriate to change the parameterization of the curve from λ to τ. Since

$$\sqrt{\dot{x}\cdot\dot{x}} = \frac{d\tau}{d\lambda},$$

we may rewrite the above as

$$\frac{d^2x^\mu}{d\tau^2} = 0,$$

or

$$u^\mu = \frac{dx^\mu}{d\tau} = \text{constant}\,.$$

That is, the world-line is a curve of constant velocity—a straight line.

Next we must show that the extremum is a *maximum*. To do this we look at the second-order variation,

$$\delta T^{(2)} = \frac{1}{2}\int_A^B d\lambda\, \frac{\dot{x}\cdot\dot{x}\;\dot{\xi}\cdot\dot{\xi} - (\dot{x}\cdot\dot{\xi})^2}{\left(\frac{d\tau}{d\lambda}\right)^3} \equiv \frac{1}{2}\int_A^B d\tau \left[\frac{d\xi}{d\tau}\cdot\frac{d\xi}{d\tau} - \left(u\cdot\frac{d\xi}{d\tau}\right)^2\right].$$

Now, since this term may be evaluated in any frame, we choose the one in which the components of $u^\mu = \frac{dx^\mu}{d\tau}$ are $\begin{pmatrix}1\\ \vec{u}=0\end{pmatrix}$ (we note that in any frame, $u\cdot u = 1$). Hence

$$\frac{d\xi}{d\tau}\cdot\frac{d\xi}{d\tau} - \left(u\cdot\frac{d\xi}{d\tau}\right)^2 = -\frac{d\vec{\xi}}{d\tau}\cdot\frac{d\vec{\xi}}{d\tau} < 0$$

hence, for any function $\xi^\mu(\tau)$ the second-order variation $\delta T^{(2)}$ is negative-definite. But this is the condition for a maximum, *i.e.* any deviation from the maximizing function

$$x^\mu(\tau) = x^\mu(0) + u^\mu\,\tau$$

leads to a *diminution* of T.

2. A collection of free particles has 4-momenta $p(i)$, i=1, … , N in the lab. Show that the square of their total 4-momentum,

$$P^\mu P_\mu = \left(\sum_{i=1}^{N} p^\mu(i)\right)\left(\sum_{i=1}^{N} p_\mu(i)\right)$$

is a Lorentz-invariant quantity, equal to the square of the total energy in a frame where the total 3-momentum vanishes. Find the Lorentz transformation connecting this "barycentric" frame with the lab frame.

Solution:

As we saw in class, the contraction of any 4-vector with any other 4-vector (such as itself!) is a Lorentz scalar. But it is also true that the sum of two 4-vectors is a 4-vector, hence by induction the sum of any number of vectors is a vector. This proves the first statement.

Consider a frame moving with velocity $\vec{v}$ relative to the one in which we have measured the above total 4-mommentum. In this frame (units are $c = 1$),

$$P'^0 = \gamma\left(P^0 - \vec{v}\cdot\vec{P}\right)$$

$$\vec{P}' = \vec{P} + (\gamma - 1)\,\hat{v}\left(\vec{P}\cdot\hat{v}\right) - \gamma\vec{v}\,P^0 .$$

Setting $\vec{P}'$ to 0 gives

$$v = \frac{\vec{P}\cdot\hat{v}}{P^0}, \quad \hat{v} = \hat{P}$$

i.e.,

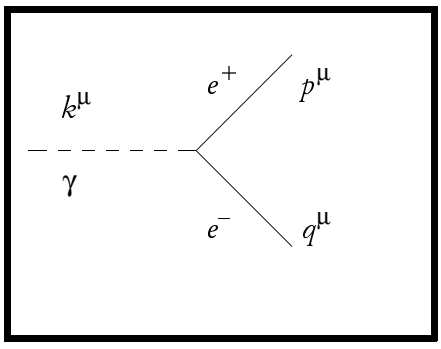
$$\vec{v} = \frac{\vec{P}}{P^0}.$$

In this frame,

$$P'^0 = \gamma\left(P^0 - \vec{v}\cdot\vec{P}\right) = \left(P^0 - \frac{\vec{P}\cdot\vec{P}}{P^0}\right)\left(1 - \frac{\vec{P}\cdot\vec{P}}{\left(P^0\right)^2}\right)^{-1/2} = \left[\left(P^0\right)^2 - \vec{P}\cdot\vec{P}\right]^{1/2},$$

as it should.

3. Explain why a photon cannot spontaneously turn into an electron-positron pair ($m_e c^2 = 0.511$ MeV).



Solution:

4-momentum conservation requires

$$k\cdot k = (p+q)\cdot(p+q) = 0$$

(because a photon has no mass, the square of its 4-momentum vanishes).

Expanding the r.h.s. we find (since $p \cdot p = q \cdot q = m^2$)

$$2m^2 + 2p \cdot q = 2m^2 + 2p^0 q^0 - 2\vec{p} \cdot \vec{q} = 0 .$$

But

$$p^0 q^0 - \vec{p} \cdot \vec{q} = \sqrt{\vec{p}^2 + m^2}\,\sqrt{\vec{q}^2 + m^2} - \vec{p} \cdot \vec{q} > 0 ;$$

hence 4-momentum conservation can never be satisfied by the decay of a photon into an electron-positron pair. And of course, the inverse process $e^+ e^- \to \gamma$ is equally forbidden—positron annihilation requires the emission of at least 2 photons.

4. A photon collides with a heavy nucleus of mass M and produces an $e^+ e^-$ pair ($m_e << M$). What is the minimum ("threshold") energy the photon must have to make this process occur?

Solution:

We apply the method used above: transform to a frame (the "barycentric" frame) where the total 3-momentum vanishes. In the laboratory the 4-momentum is the sum of that for the photon and that of the stationary target:

$$\begin{pmatrix} k^0 \\ \vec{k} \end{pmatrix} + \begin{pmatrix} M \\ \vec{0} \end{pmatrix} = \begin{pmatrix} k^0 + M \\ \vec{k} \end{pmatrix}$$

In the barycentric frame the total 4-momentum is

$$\begin{pmatrix} \sqrt{\vec{p}\,'^2 + m^2} + \sqrt{\vec{q}\,'^2 + m^2} + \sqrt{(-\vec{p}\,' - \vec{q}\,')^2 + M^2} \\ \vec{0} \end{pmatrix}$$

Setting the contractions of these with themselves ("invariant masses"-squared) equal, we have

$$\left(k^0\right)^2 + M^2 + 2k^0 M - \vec{k}^2 = \left(\sqrt{\vec{p}\,'^2 + m^2} + \sqrt{\vec{q}\,'^2 + m^2} + \sqrt{(-\vec{p}\,' - \vec{q}\,')^2 + M^2}\right)^2$$

or since

$$\left(k^0\right)^2 - \vec{k}^2 = 0 ,$$

$$2k^0 M = \left(\sqrt{\vec{p}\,'^2 + m^2} + \sqrt{\vec{q}\,'^2 + m^2} + \sqrt{(\vec{p}\,' + \vec{q}\,')^2 + M^2}\right)^2 - M^2$$

The r.h.s. is obviously minimized by setting

$$\vec{p}\,' + \vec{q}\,' = 0 ,$$

and then

$$\vec{p}\,' = \vec{q}\,' = 0 ,$$

giving

$$2k^0_{\min} M = \left(2m + M\right)^2 - M^2 = 4m^2 + 4mM ,$$

$$k^0_{\min} = 2m\left(1 + m/M\right).$$

5. A collision between a proton with kinetic energy T and a proton that is stationary in the laboratory frame produces the charmed baryon-antibaryon pair $\Lambda_c\,\overline{\Lambda}_c$ in the reaction

$$p + p \rightarrow p + p + \Lambda_c + \overline{\Lambda}_c\,.$$

The mass of the proton is 938 MeV, and that of the Λ_c (and $\overline{\Lambda}_c$) is 2281 MeV. What is the minimum (laboratory) kinetic energy T needed to produce the $\Lambda_c\,\overline{\Lambda}_c$ pair?

Solution:

Following what we did in Problem 4 we can say immediately

$$\left(p^0 + m\right)^2 - \vec{p}^2 = \left(2m + 2M\right)^2;$$

using

$$\left(p^0\right)^2 - \vec{p}^2 = m^2$$

we find

$$2p^0 m + 2m^2 = 4\left(m^2 + 2mM + M^2\right)$$

or

$$p^0 m \equiv \left(T + m\right)m = m^2 + 4mM + 2M^2$$

and

$$T = 4M\left(1 + \frac{M}{2m}\right) = 4 \times 2281 \times \left(1 + \frac{2281}{2 \times 938}\right) = 20{,}218 \text{ MeV}\,.$$