HW #3—Solutions

Ohanian, Chapter 2

18. Consider two inertial frames moving relative to each other, and take the x-axis along the direction of motion. Let us integrate $J^\mu(x)\,\partial_\mu\,\theta\left(n_\beta\,x^\beta\right)$ over a 4-dimensional volume, where the vector n_β has the components $(1, 0, 0, 0)$ in the lab frame, and the unit step function is

$$\theta(s) = \begin{cases} 1, & s > 0 \\ 0, & s \le 0 \end{cases}.$$

The total charge may then be written

$$Q = \iiint\limits_{\Omega} d^4x\, J^\mu(x)\,\partial_\mu\,\theta\left(n_\beta\,x^\beta\right) = \iiint\limits_{V} d^3x \int dt\, J^\mu(x)\,\partial_\mu\,\theta(t) = \iiint\limits_{V} d^3x \int dt\, J^0(x)\,\delta(t)\,.$$

Manifestly, both sides of the above equation are scalars, because the Jacobian of the transformed coordinates is

$$J = \det\begin{vmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial y} & \frac{\partial t'}{\partial z} \\ \frac{\partial x'}{\partial t} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial t} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{vmatrix} = \det\begin{vmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \gamma^2\left(1 - v^2\right) \equiv 1\,,$$

giving

$$d^4x' = J\,d^4x = d^4x\,.$$

We might then ask, "Why isn't this enough?" That is, we have managed to write the total charge at some arbitrary time $\widetilde{t}$ (in the above case $\widetilde{t}$ was 0, but we could have subtracted $\widetilde{t}$ from the argument of the θ-function to evaluate at any time) as a Lorentz scalar quantity. The problem is that if we worked in a different inertial frame, n_β would no longer have the simple form given above (point in the time direction) but would be—say—

$$n'_\beta = \begin{pmatrix} \gamma \\ v\gamma \\ 0 \\ 0 \end{pmatrix}.$$

This would give a different, and somewhat more complicated form for Q' which would not be obviously equal to Q. However, if we write

$$Q - Q' = \iiint\limits_{\Omega} d^4x\, J^\mu(x)\,\partial_\mu\left[\theta\left(n_\beta x^\beta\right) - \theta\left(n'_\beta\,x^\beta\right)\right],$$

we can integrate by parts and apply the 4-dimensional Gauss theorem:

$$\iiiint_\Omega d^4x\, J^\mu(x)\, \partial_\mu\, \varphi(x) \;=\; \iiiint_\Omega d^4x\, \partial_\mu \Big[J^\mu(x)\, \varphi(x) \Big] \;-\; \iiiint_\Omega d^4x\, \varphi(x)\, \partial_\mu J^\mu(x)$$

If the current is conserved, $\partial_\mu J^\mu(x) = 0$, the second term vanishes and the first term, being a pure divergence, can be rewritten as an integral over a (closed) hypersurface:

$$\iiiint_\Omega d^4x\, \partial_\mu \Big[J^\mu(x)\, \varphi(x) \Big] = \iiint_\Sigma d\sigma_\mu J^\mu(x)\, \varphi(x)\,.$$

Since $J^\mu(x) \;\to\; 0$ as $|\,\vec{x}\,| \to \infty$ and $\varphi(x) \equiv \theta(t) \;-\; \theta(\gamma t + v\gamma x) \;\to\; 0$ as $|t| \to \infty$, we see the right side of the above integral vanishes, giving $Q = Q' =$constant; and moreover, showing that Q is a Lorentz scalar.

23. We see that

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 = -\partial_t A^1 - \partial_1 A^0 = E^1$$

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = -\partial_1 A^2 + \partial_2 A^1 = -B^3\,.$$

from these we generalize to

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

To calculate the contraction $F_{\mu\nu} F^{\mu\nu}$ we take the transpose of the second matrix and multiply them, then take the trace. The matrix product is

$$\begin{pmatrix} -\vec{E}^2 & ?? & ?? & ?? \\ ?? & (B^2)^2 + (B^3)^2 - (E^1)^2 & ?? & ?? \\ ?? & ?? & (B^1)^2 + (B^3)^2 - (E^2)^2 & ?? \\ ?? & ?? & ?? & (B^1)^2 + (B^2)^2 - (E^3)^2 \end{pmatrix}$$

where we have not bothered to compute the off-diagonal elements because they contribute nothing to the trace. Taking the trace we obtain $2\left(\vec{B}^2 \;-\; \vec{E}^2\right)$, as advertised.

24. The electromagnetic field in the lab is $E^3 \equiv -E_z = \dfrac{-2\lambda}{z}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E^3 & 0 & 0 & 0 \end{pmatrix}$$

We now Lorentz transform on each of the indices:

$$F'^{\mu\nu} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -E^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E^3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\gamma E^3 \\ 0 & 0 & 0 & v\gamma E^3 \\ 0 & 0 & 0 & 0 \\ \gamma E^3 & -v\gamma E^3 & 0 & 0 \end{pmatrix}.$$

By inspection we can then identify $B'^2 = v\gamma E^3$, $E'^3 = \gamma E^3$.

Extra problems:

1. Using the formulas for (Lorentz) transformation of velocity show that the relativistic definitions of energy and 3-momentum,

$$E = \frac{mc^2}{\sqrt{1 - \vec{u}^2/c^2}}$$

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \vec{u}^2/c^2}}$$

indeed transform as the components of a 4-vector.

Solution:

Assume the particle has velocity components parallel and perpendicular to the relative velocity between frames—call them u_x and u_y, respectively. The velocity transformation formulas then are

$$u'_x = \frac{u_x - V}{1 - Vu_x}$$

$$u'_y = \frac{1}{\gamma}\frac{u_y}{1 - Vu_x}$$

where $\dfrac{1}{\gamma} = \sqrt{1 - V^2}$.

Now, in the S' system the energy and momentum are

$$E' = \frac{mc^2}{\sqrt{1-\vec{u}'^{\,2}/c^2}}$$

$$\vec{p}' = \frac{m\vec{u}'}{\sqrt{1-\vec{u}'^{\,2}/c^2}}$$

letting $c=1$ to simplify the formulas, we see that

$$\sqrt{1-\vec{u}'^{\,2}} = \left(\frac{\sqrt{1-V^2}}{1-Vu_x}\right)\sqrt{1-\vec{u}^2}$$

leading us to

$$p'_y = \frac{mu'_y\,(1-Vu_x)}{\sqrt{(1-V^2)\,(1-u^2)}} = \frac{mu_y}{\sqrt{(1-u^2)}} = p_y$$

$$p'_x = \frac{mu'_x\,(1-Vu_x)}{\sqrt{(1-V^2)\,(1-u^2)}} = \frac{m\,(u_x-V)}{\sqrt{(1-V^2)\,(1-u^2)}} = \gamma\left(p_x - VE\right)$$

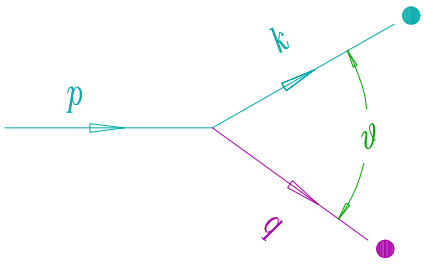
$$E' = \frac{m\,(1-Vu_x)}{\sqrt{(1-V^2)\,(1-u^2)}} = \gamma\left(E - Vp_x\right).$$

QED.

2. A particle collides elastically with a stationary particle of the same mass. Show that the angle between the trajectories of the outgoing particles is smaller than $\frac{\pi}{2}$, approaching this value in the limit of nonrelativistic velocities.

Solution:

Four-momentum conservation—as shown to the right—gives



$$\begin{pmatrix}\sqrt{p^2+m^2}+m\\ \vec{p}\end{pmatrix} = \begin{pmatrix}\sqrt{k^2+m^2}+\sqrt{q^2+m^2}\\ \vec{k}+\vec{q}\end{pmatrix}.$$

On calculating the invariant mass (Lorentz-invariant square of the momentum on both sides and setting them equal, we have

$$p^2+m^2+2m\sqrt{p^2+m^2}+m^2-\vec{p}^2$$

$$= k^2+m^2+q^2+m^2+2\sqrt{k^2+m^2}\sqrt{q^2+m^2}-\vec{k}^2-\vec{q}^2-2\vec{k}\cdot\vec{q}\ .$$

Cancelling out various terms and substituting $\vec{k}+\vec{q}$ for $\vec{p}$ under the radical sign we can solve for

$$\vec{k} \cdot \vec{q} = kq \cos\theta$$

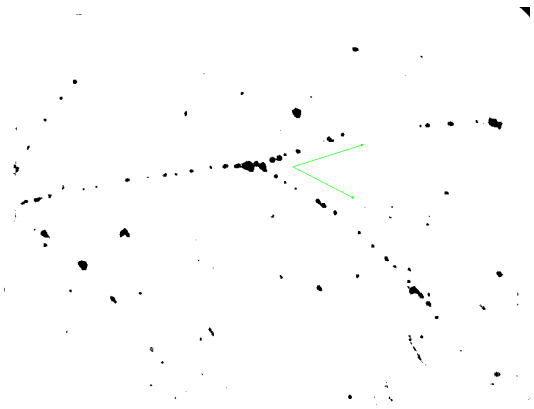
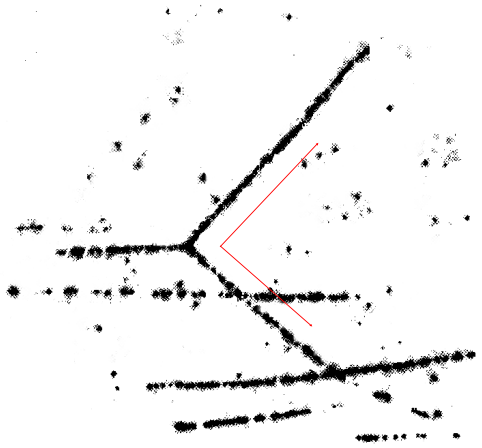
We can then solve the resulting quadratic equation to find

$$\cos\theta = w \pm \sqrt{w^2 - 1}$$

where

$$w = \frac{m^2 + \sqrt{k^2 + m^2}\ \sqrt{q^2 + m^2}}{qk} \equiv \frac{m^2}{qk} + \sqrt{1 + m^2/k^2}\ \sqrt{1 + m^2/q^2} > 1 .$$

Because w is greater than 1, only the root

$$\cos\theta = w - \sqrt{w^2 - 1} \equiv \frac{1}{w + \sqrt{w^2 - 1}} < 1$$

is physical. However it is worth noting that if $p \ll m$, $\cos\theta \to 0$, *i.e.* in that limit the scattering angle approaches 90^o. But since $0 < \cos\theta < 1$, the scattering angle must always be less than 90^o.