HW #4—Solutions

Ohanian, Chapter 2

26. First of all, we see that the given fields satisfy the Maxwell equations in vacuum—it's always wise to check! The energy-momentum tensor (Ohanian, eq. 127) is

$$T^{\mu\nu} = \frac{1}{16\pi}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{4\pi}F^{\mu\alpha}F^{\nu\beta}\eta_{\alpha\beta};$$

the first term is proportional to $\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E}$, hence it vanishes. The second term is proportional to the matrix product

$$-\begin{pmatrix}0&-1&0&0\\1&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0&0\\-1&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}$$

hence the components of the tensor are

$$T^{\mu\nu} = \frac{E_0^2}{4\pi}\cos^2\Big(\omega(t-z)\Big)\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}.$$

Obviously the energy flux (energy density times velocity—note $c=1$ in this system of units) equals the component T^{03}. For a gas of photons, the momentum of each photon is the same as its energy—thus the energy density times velocity is the same as the momentum flux. Also note that the Poynting vector and energy density are equal in magnitude, as they should be for photons.

28. We have already seen that for a gas the energy-momentum tensor has the form

$$T^{00} = \rho,\ T^{0k} = 0,\ T^{ab} = p\,\delta^{ab};$$

we are thus required to boost it to a frame in which the total momentum is non-vanishing. That is,

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}$$

or

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0}T^{00} + \sum_{k=1}^{3}\Lambda^{\mu}{}_{k}\Lambda^{\nu}{}_{k}T^{kk} \equiv \rho\,\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} + p\sum_{k=1}^{3}\Lambda^{\mu}{}_{k}\Lambda^{\nu}{}_{k}.$$

We may simplify the calculation if we pick the direction of relative motion to be the 1-axis (x-axis):

$$\Lambda = \begin{pmatrix} \gamma & V\gamma & 0 & 0 \\ V\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} U^0 & U^1 & 0 & 0 \\ U^1 & U^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here we have used

$$U^0 = \frac{dt}{d\tau} = \gamma$$

$$U^1 = \frac{dx^1}{d\tau} = \frac{dx^1}{dt}\frac{dt}{d\tau} = V\gamma$$

$$U^2 = U^3 = 0 \ .$$

The components of T' are thus

$$T' = \begin{pmatrix} (\rho+p)U^0U^0 - p & (\rho+p)U^0U^1 & 0 & 0 \\ (\rho+p)U^1U^0 & (\rho+p)U^1U^1 - p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = (\rho+p)\, U^\mu U^\nu - p\, \eta^{\mu\nu} \ ;$$

to obtain this we have used the relation $U^\mu U_\mu \equiv \left(U^0\right)^2 - \left(U^1\right)^2 = 1$.

Extra problem:

1. The relativistic Lagrangian of a free particle is

$$L = -mc^2\sqrt{1 - \vec{u}\cdot\vec{u}/c^2} \ .$$

a) Use this to find an expression for the 3-momentum of the particle *via* $\vec{p} = \dfrac{\partial L}{\partial \vec{u}}$.

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = -mc^2 \times \frac{1}{2} \times \frac{-2\vec{u}/c^2}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}} = \frac{m\vec{u}}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}}$$

b) The Hamiltonian (energy) is defined as $H = \vec{u}\cdot\vec{p} - L$. Hence find an explicit expression for the total energy.

$$H = \vec{u}\cdot\vec{p} - L = mc^2\left(\frac{\vec{u}\cdot\vec{u}/c^2}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}} + \sqrt{1 - \vec{u}\cdot\vec{u}/c^2}\ \right) = \frac{mc^2}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}}$$

c) Do the above expressions for momentum and energy agree with what we found before? *Mais, oui!*

2. Suppose the above particle finds itself in external central potentials: $V(r)$ that resembles the time component of a 4-vector; and $S(r)$ that behaves like a scalar quantity (it combines with the mass); so that the Lagrangian becomes

$$L = -\left(mc^2 + S(r)\right)\sqrt{1 - \vec{u}\cdot\vec{u}/c^2} - V(r) .$$

What are the resulting 3-momentum, energy and equation of motion? From Newton's Law identify the force on the particle.

As before, defining $\vec{p} = \dfrac{\partial L}{\partial \vec{u}}$ we see that

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = \left(m + S(r)/c^2\right)\frac{\vec{u}}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}} .$$

The corresponding energy is

$$H = \vec{u}\cdot\vec{p} - L = \frac{\left(mc^2 + S(r)\right)}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}} + V(r)$$

Finally, the equation of motion is

$$\frac{d\vec{p}}{dt} \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{u}}\right) = \frac{\partial L}{\partial \vec{x}} \equiv \nabla L = -\nabla S(r)\sqrt{1 - \vec{u}\cdot\vec{u}/c^2} - \nabla V(r)$$

i.e.,

$$\vec{F} = -\nabla S(r)\sqrt{1 - \vec{u}\cdot\vec{u}/c^2} - \nabla V(r)$$

This tells us that a force arising from a scalar potential gets weaker with increasing velocity, and one arising from a vector potential does not vary with velocity. Thus, if we attempt to generalize gravitation to a Lorentz scalar (in order to make it relativistic), a hot gas would weigh *less* than a cold one (particles moving faster, on average), which is at variance with experience (Eötvos experiment). Similarly, if we try to make gravity a vector (by analogy with electromagnetism), we find hot and cold bodies weigh the same, again in violation of experiment. However if we add the 00 component of a tensor field to the Lagrangian, it must have the form

$$L = -mc^2\sqrt{1 - \vec{u}\cdot\vec{u}/c^2} - m\frac{h^{00}(r)}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}}$$

giving a gravitational force that is proportional to the energy content of an object:

$$\vec{F} = -\nabla\left(m\frac{h^{00}(r)}{\sqrt{1 - \vec{u}\cdot\vec{u}/c^2}}\right).$$