HW #5—Solutions

1. Manifestly, because all indices are dummy indices,

$$\partial_\mu \widetilde{F}^{\mu\nu} = \partial_\mu \varepsilon^{\mu\nu\kappa\lambda}\left(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa\right)$$
$$\equiv 2\,\varepsilon^{\mu\nu\kappa\lambda}\,\partial_\mu\partial_\kappa A_\lambda = \varepsilon^{\mu\nu\kappa\lambda}\left(\partial_\mu\partial_\kappa - \partial_\kappa\partial_\mu\right)A_\lambda = 0\ .$$

2. It is easy to see that since

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \quad \widetilde{F}^{\mu\nu} = 2\begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

$$\widetilde{F}^{\mu\nu} F_{\mu\nu} = -8\,\vec{E}\cdot\vec{B}.$$

Hence, because $\vec{E} = -\nabla A^0 - \partial_t \vec{A}$ is an ordinary vector under reflections, $\vec{B} = \nabla\times\vec{A}$ must be a polar (pseudo-) vector. That is, they have opposite parity, hence $\widetilde{F}^{\mu\nu} F_{\mu\nu} \underset{\vec{x}\to-\vec{x}}{\longrightarrow} -\widetilde{F}^{\mu\nu} F_{\mu\nu}$. In other words, $\widetilde{F}$ and F transform oppositely under reflections.

3. We can write

$$\mathscr{H} = \partial_0\varphi\left(\frac{\partial\mathscr{L}}{\partial\varphi_{,0}}\right) - \mathscr{L} \equiv \partial_0\varphi\left(\frac{\partial\mathscr{L}}{\partial\varphi_{,0}}\right) - \eta^{00}\mathscr{L}$$

or in other words,

$$\mathscr{T}^{\mu\nu} = \tfrac{1}{2}\partial^\mu\varphi\left(\frac{\partial\mathscr{L}}{\partial\varphi_{,\nu}}\right) + \tfrac{1}{2}\partial^\nu\varphi\left(\frac{\partial\mathscr{L}}{\partial\varphi_{,\mu}}\right) - \eta^{\mu\nu}\mathscr{L}$$

If we take the divergence of the above expression we obtain

$$\partial_\mu\mathscr{T}^{\mu\nu} = \tfrac{1}{2}\partial_\mu\left[\varphi,^\nu W^\mu - \varphi,^\mu W^\nu\right], \quad W^\mu = \frac{\partial\mathscr{L}}{\partial\varphi_{,\mu}}$$

The crucial fact, however, is that since $\varphi_{,\mu}$ appears in the Lagrangian density quadratically, *i.e.* in the form

$$\varphi_{,\mu}\,\varphi^{,\mu} \times \text{Lorentz scalar},$$

it must be true then that $W^{\mu} \propto \varphi^{,\mu}$, therefore

$$\partial_{\mu}\mathcal{T}^{\mu\nu} = \tfrac{1}{2}\partial_{\mu}\left\{\left[\varphi^{,\nu}\varphi^{,\mu} - \varphi^{,\mu}\varphi^{,\nu}\right] \times \text{Lorentz scalar}\right\} \equiv 0$$

4. The Lagrangian has the form

$$\mathcal{L} = \frac{-1}{8\pi}\left(\vec{B}^2 - \vec{E}^2\right) + \frac{1}{8\pi}\kappa^2\left(\left(A^0\right)^2 - \left(\vec{A}\right)^2\right) - J^0A^0 + \vec{J}\cdot\vec{A}$$

and in the absence of currents (and therefore of vector potential) becomes

$$\mathcal{L} = \frac{1}{8\pi}\left(\nabla A^0 \cdot \nabla A^0 + \kappa^2\left(A^0\right)^2\right) - J^0A^0$$

The Euler-Lagrange equation is then

$$\frac{1}{4\pi}\left(\nabla^2 A^0 - \kappa^2 A^0\right) + J^0 = 0,$$

giving, for a point charge at the origin, $J^0 = Q\,\delta(\vec{x})$, the required solution,

$$A^0 = \frac{Q\,e^{-\kappa r}}{r}.$$