HW #6 Solutions

1. Parabolic coordinates are defined by

$$x = (\xi\eta)^{1/2}\cos\varphi$$

$$y = (\xi\eta)^{1/2}\sin\varphi$$

$$z = \frac{1}{2}(\xi - \eta)$$

a. Give an expression for $r = \left(x^2 + y^2 + z^2\right)^{1/2}$ in terms of ξ, η, and φ ; and expressions for ξ, η, and φ in terms of r, x, y, z.

b. Find an expression for the element of length in terms of ξ, η and φ . Are these coordinates orthogonal?

c. Find expressions for $g_{\mu\nu}$ and $g^{\mu\nu}$.

d. Express the Laplacian operator ∇^2 in terms of ξ, η and φ .

Solution:

a) $r^2 = x^2 + y^2 + z^2 = \xi\eta + \frac{1}{4}(\xi - \eta)^2 = \frac{1}{4}(\xi + \eta)^2$, $r = \frac{1}{2}(\xi + \eta)$.

Obviously,

$$\xi = r + z$$

$$\eta = r - z$$

$$\varphi = \tan^{-1}\left(y/x\right)$$

b) By definition,

$$d\vec{x}\cdot d\vec{x} = \left(dx\right)^2 + \left(dy\right)^2 + \left(dz\right)^2$$

$$= \left(\frac{1}{2}\left[(\eta/\xi)^{1/2}\,d\xi + (\xi/\eta)^{1/2}\,d\eta\right]\cos\varphi - (\xi\eta)^{1/2}\sin\varphi\;d\varphi\right)^2 +$$

$$= \left(\frac{1}{2}\left[(\eta/\xi)^{1/2}\,d\xi + (\xi/\eta)^{1/2}\,d\eta\right]\sin\varphi + (\xi\eta)^{1/2}\cos\varphi\;d\varphi\right)^2 + \frac{1}{4}\left(d\xi - d\eta\right)^2$$

$$= \frac{1}{2}\left(r/\xi\right)\left(d\xi\right)^2 + \frac{1}{2}\left(r/\eta\right)\left(d\eta\right)^2 + \xi\eta\left(d\varphi\right)^2 .$$

Since there are no cross-terms, the coordinates are orthogonal.

c) $g_{\xi\xi} = \frac{r}{2\xi}$, $g_{\eta\eta} = \frac{r}{2\eta}$, $g_{\varphi\varphi} = \xi\eta$.

Clearly,

$g^{\xi\xi} = 1/g_{\xi\xi}$, *etc.*

d) $g = \det\left(g_{\mu\nu}\right) = \frac{1}{4} r^2$

$$\nabla^2 \Phi = g^{-1/2} \frac{\partial}{\partial x^k}\left(g^{1/2}\, g^{kl} \frac{\partial \Phi}{\partial x^l} \right)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial \xi}\left(r g^{\xi\xi} \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(r g^{\eta\eta} \frac{\partial \Phi}{\partial \eta} \right) + \frac{\partial}{\partial \varphi}\left(r g^{\varphi\varphi} \frac{\partial \Phi}{\partial \varphi} \right) \right]$$

$$= \frac{4}{\xi + \eta}\left[\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \Phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta}\left(\eta \frac{\partial \Phi}{\partial \eta} \right) \right] + \frac{1}{\eta\xi} \frac{\partial^2 \Phi}{\partial \varphi^2} .$$

2. Write the energy-momentum tensor for a rapidly rotating uniform distribution of mass. Use two methods:

a) the definition of $T^{\mu\nu}$ for a collection of point particles;

b) the definition for a static mass transformed to a rotating coordinate system.

Do the results differ? If so, why?

Solution:

a) The energy-momentum tensor for a collection of point particles is

$T^{\mu\nu} = U^\mu U^\nu (\rho + p) - \eta^{\mu\nu} p$.

In a rotating coordinate system, the local (average) 3-velocity of a particle at position $\vec{r}$ is

$\vec{u} = \vec{\omega} \times \vec{r}$.

The 4-velocity is therefore

$U^\mu = \left(\gamma,\ \gamma\, \vec{\omega} \times \vec{r}\, / c \right)$

where

$\gamma = \left[1 - \left(\vec{\omega} \times \vec{r} \right)^2 / c^2 \right]^{-1/2}$.

Assume p may be neglected compared with ρ: then

$T^{00} = \gamma^2 \rho$

$T^{0k} = \gamma^2 \rho \left(\vec{\omega} \times \vec{r} \right)^k / c$

$T^{kl} = \gamma^2 \rho \left(\vec{\omega} \times \vec{r} \right)^k \left(\vec{\omega} \times \vec{r} \right)^l / c^2$

b) The energy-momentum tensor of a static mass is $T'^{00} \approx \rho$. Now transform from a rotating coordinate system to the lab: this is equivalent to transforming to a system rotating with angular velocity $\vec{\omega} = -\vec{\omega}'$, relative to the body-fixed system. That is, the instantaneous 3-velocity of a point at position $\vec{r}'$ in the body is $\vec{v}' = -\vec{\omega}' \times \vec{r}'$ in the lab. The Lorentz transformation between these coordinates is then

$$\vec{r}_\perp = \vec{r}' - \hat{v}' \hat{v}' \cdot \vec{r}' \equiv \vec{r}'$$

(since $\vec{r}'$ is perpendicular to the velocity),

$$\vec{r}_\parallel = \gamma\left(\vec{r}'_\parallel - \vec{v}' t\right) \equiv -\gamma \vec{v}' t' ,$$

and

$$t = \gamma t' .$$

We also note

$$\frac{\partial t}{\partial t'} = \gamma + t' \frac{\partial \gamma}{\partial t'},$$

$$\frac{\partial}{\partial t'} \vec{r}_\parallel \equiv -\gamma \vec{v}' - \vec{v}' t' \frac{\partial \gamma}{\partial t} .$$

If the angular velocity is constant, then $\frac{\partial \gamma}{\partial t'}$. The lab energy-momentum tensor is thus

$$T^{00} = T'^{00}\left(\frac{\partial t}{\partial t'}\right)^2 = \gamma^2 \rho$$

$$T^{0k} = T'^{00} \frac{\partial t}{\partial t'} \frac{\partial \vec{x}}{\partial t'} = \gamma^2 \rho \left(\vec{\omega} \times \vec{r}\right)^k / c$$

$$T^{kl} = T'^{00} \frac{\partial x^k}{\partial t'} \frac{\partial x^l}{\partial t'} = \gamma^2 \rho \left(\vec{\omega} \times \vec{r}\right)^k \left(\vec{\omega} \times \vec{r}\right)^l / c^2$$

as before. Are they the same? Not identically—in the first case we neglected the pressure, which is a measure of internal stress. In general, a physical object has internal forces that make it cohere, but which are negligible relative to its energy density. For certain kinds of object, notably neutron stars, this is no longer true, and the internal forces must be accounted for in a general coordinate transformation.

3. Defining the intrinsic spin associated with $T^{\mu\nu}(x)$ as

$$S^j = \sum_{k,l} \varepsilon^{jkl} \int d^3x \, T^{0k}(x) \, x^l$$

express the gravitational field $h^{\mu\nu}$ in terms of $\vec{S}$ and M.

Solution:

For a time-independent source,

$$\zeta^{\mu\nu}(\vec{r}) = -2G\int d^3x\,\frac{T^{\mu\nu}(\vec{x})}{|\vec{r}-\vec{x}|}.$$

Clearly, for a spherically symmetric spinning object,

$$\int d^3x\, T^{\mu\nu}(\vec{x}) \approx \begin{cases} M, & \mu=\nu=0 \\ 0, & \text{otherwise} \end{cases}$$

The next term comes from

$$\int d^3x\, T^{\mu\nu}(\vec{x})\,\vec{x}\cdot\hat{r} = \int d^3x\, T^{0k}(\vec{x})\,\vec{x}\cdot\hat{r}, \text{ all others } 0.$$

Hence,

$$\int d^3x\, T^{0k}(\vec{x})\,\vec{x}\cdot\hat{r} = \sum_j \frac{r^j}{r}\int d^3x\, T^{0k}(\vec{x})\,x^j \equiv \sum_{j,l}\frac{r^j}{r}\,S^l\,\varepsilon^{ljk} = \frac{1}{r}\left(\vec{S}\times\vec{r}\right)^k.$$

Hence

$$\zeta^{00}(\vec{r}) \approx \frac{-2GM}{r}$$

$$\zeta^{0k}(\vec{r}) \approx \frac{-2GM}{r^3}\left(\vec{S}\times\vec{r}\right)^k$$

4. Express the deflection of a light ray in the field of a rotating mass, assuming the ray moves in the equatorial plane of the mass.

Solution:

The easiest method employs the equation of motion of a test particle in the linearized approximation:

$$\frac{dU^\mu}{d\tau} + \left(h_{\mu\lambda,\kappa} - \tfrac{1}{2}h_{\lambda\kappa,\mu}\right)U^\lambda U^\kappa = 0$$

Treating the problem as one of small-angle deflection, we have the initial 4-velocity of a particle moving with 3-velocity $v\hat{x}$:

$$U^\mu = \begin{pmatrix}\gamma\\ \gamma v\\ 0\\ 0\end{pmatrix}$$

The final 4-velocity is

$$U'^\mu = U^\mu + \Delta U^\mu = U^\mu - \int_{-\infty}^{+\infty} d\tau\left(h_{\mu\lambda,\kappa} - \tfrac{1}{2}h_{\lambda\kappa,\mu}\right)U^\lambda(\tau)\,U^\kappa(\tau);$$

assuming small deflection,

$$U^\lambda(\tau) \approx U^\lambda(-\infty) = U^\lambda,$$

and scattering in the equatorial plane, $z(\tau) = z(-\infty) = 0$, we find

$$U'^0 = U^0 = \gamma,$$

$$U'^{z} = 0\,,$$

$$U'^{y} = -\frac{1}{v\gamma}\int_{-\infty}^{+\infty} dx\left[\left(h_{y0,x} - h_{0x,y}\right)v - \frac{1}{2}h_{00,y}\left(1 + v^2\right)\right]$$

Note: $h_{00} = h_{xx}$

Since

$$h^{00} = \zeta^{00} = \frac{-2MG}{r},$$

$$h^{0k} = \zeta^{0k}(\vec{r}\,) \approx \frac{-2GM}{r^3}\left(\vec{S}\times\vec{r}\right)^{k}$$

with $\vec{S} = S\hat{z}$

$$h_{0x} = \frac{-2GSy}{r^3}, \qquad h_{0y} = \frac{2GSx}{r^3},$$

we find

$$h_{y0,x} - h_{0x,y} = \frac{-2GS}{r^3}\,.$$

The deflection angle is thus (b is the impact parameter)

$$\theta \approx \frac{U'^{y}}{U^{x}} = \frac{-2GM\,(1+v^2)}{bv^2} + \frac{4GS}{b^2v^2}\,\mathrm{sgn}(y)\,.$$

That is, it matters which side of the spinning object the light passes on.

5. Find a Lagrangian density that yields the (linearized) gravitational field equations. From this, evaluate the energy-momentum tensor $t_{(1)}^{\mu\nu}$ of the gravitational field, in the weak-field approxiation.

Solution:

$$\partial_\lambda\,\partial^\lambda\,\varphi^{\mu\nu} + KT^{\mu\nu} = 0\,.$$

Since the energy momentum tensor for a point particle is

$$\int d\tau\,\delta^{(4)}\left(x - \xi(\tau)\right)m\,\frac{d\xi^{\mu}(\tau)}{d\tau}\,\frac{d\xi^{\nu}(\tau)}{d\tau}\,,$$

we clearly want

$$\mathscr{L} = T^{\mu\nu}\,\varphi_{\mu\nu} + \left(\text{quadratic expression in } \varphi_{\mu\nu\,,\,\lambda}\right).$$

That is,

$$\mathscr{L} = T^{\mu\nu}\,\varphi_{\mu\nu} - \frac{1}{2K}\left(\varphi_{\mu\nu\,,\,\lambda}\,\varphi^{\mu\nu\,,\,\lambda}\right).$$

Check:

$$\frac{\partial}{\partial x^\lambda}\left(\frac{\partial \mathscr{L}}{\partial \varphi_{\mu\nu,\lambda}}\right) - \frac{\partial \mathscr{L}}{\partial \varphi_{\mu\nu}} = 0$$

implies

$$-\frac{1}{K}\partial_\lambda \partial^\lambda \varphi_{\mu\nu} - T^{\mu\nu} = 0\,.\ \text{ok}$$

The energy-momentum tensor of the field is therefore

$$t^{\mu\nu}_{(1)} = g^{\sigma\mu}\varphi_{\lambda\kappa,\sigma}\left(\frac{\partial \mathscr{L}}{\partial \varphi_{\lambda\kappa,\nu}}\right) - \eta^{\mu\nu}\mathscr{L}_{field} = -\frac{1}{K}\left(g^{\sigma\mu}\varphi_{\lambda\kappa,\sigma}\varphi^{\lambda\kappa,\nu} - \tfrac{1}{2}\eta^{\mu\nu}\varphi_{\lambda\kappa,\sigma}\varphi^{\lambda\kappa,\sigma}\right).$$

6. Use the result of #5 to calculate the correction to the gravitational field of a point mass due to the self-interaction of the field—that is, solve $\partial_\lambda \partial^\lambda \varphi^{\mu\nu} = -K\left(T^{\mu\nu} + t^{\mu\nu}_{(1)}\right)$.

Solution:

For a static source, $\partial_\lambda \partial^\lambda = -\nabla^2$; the source is $T^{00} = M\,\delta(\vec{r}\,)$, all others 0. Then

$$-\nabla^2 \varphi^{00}_{(0)} = -8\pi G M\,\delta(\vec{r}\,)\,,$$

all other components vanish. We must write the tensor $t^{\mu\nu}_{(1)}$ in terms of the 0'th order fields. Note the time derivatives vanish. This yields

$$-\nabla^2 \varphi^{00}_{(1)} = \frac{-2G^2M^2}{r^4}$$

$$-\nabla^2 \varphi^{kl}_{(1)} = \frac{4G^2M^2}{r^4}\left(\frac{x^k x^l}{r^2} + \frac{1}{2}\delta^{kl}\right)$$

We note that $\frac{3}{2}\frac{x^k x^l}{r^2} - \frac{1}{2}\delta^{kl}$ is a spherical harmonic of order 2, hence write

$$\varphi^{kl}_{(1)} = \left(\frac{3}{2}\frac{x^k x^l}{r^2} - \frac{1}{2}\delta^{kl}\right)f(r) + \delta^{kl}\,g(r)$$

giving

$$\frac{2\times 3}{r^2}\,f(r) - \frac{1}{r}\frac{d^2}{dr^2}\Big(rf(r)\Big) = \frac{8M^2G^2}{3r^4}$$

$$-\frac{1}{r}\frac{d^2}{dr^2}\Big(rg(r)\Big) = \frac{10M^2G^2}{3r^4}$$

$$\frac{1}{r}\frac{d^2}{dr^2}\Big(r\varphi^{00}_{(1)}(r)\Big) = \frac{2M^2G^2}{r^4}\ .$$

These can be solved easily to get

$$g(r) = -\frac{5M^2G^2}{3r^2}$$

$$\varphi^{00}_{(1)} = \frac{M^2G^2}{r^2}$$

$$f(r) = \frac{2M^2G^2}{3r^2}$$

i.e.

$$\varphi^{kl}_{(1)} = \frac{M^2G^2}{r^2}\left(\frac{x^k x^l}{r^2} - 2\delta^{kl}\right).$$