# HW #7

1. Calculate the first post-Newtonian corrections to the perihelion, $r_<$ , and aphelion, $r_>$ , in a Schwarzshild metric.

   Under what condition(s) is the orbit circular in Newtonian approximation? Does it remain circular when the general-relativistic corrections are made?

**Solution:**

We begin with the orbit equation

$$\frac{dr}{d\varphi} = \pm\left[1 - \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2E}{m} + \frac{J^2}{m^2r^2}\right)\right]^{1/2} \frac{mr^2}{J}$$

which can be converted to the form (we put in the factors of c required for dimensional consistency)

$$\frac{du}{d\varphi} = \pm\left[-u^2 + 2au - b + \frac{2MGu^3}{c^2}\right]^{1/2}$$

*via* the substitution $u = \frac{1}{r}$, where

$$a = \frac{m^2}{J^2} GM\left(1 - \frac{2E}{mc^2}\right)$$

$$b = -\frac{2Em}{J^2} .$$

For a bound orbit, $E < 0$ and $b > 0$. Factoring the cubic polynomial we have

$$-u^2 + 2au - b + 2MGu^3 = \left(u_> - u\right)\left(u - u_<\right)\left(\alpha - 2MGu\right)$$

where

$$\alpha = 1 - \frac{2MG}{c^2}\left(u_< + u_>\right)$$

$$u_> u_< = \frac{b}{\alpha}$$

$$u_> + u_< = \frac{2}{\alpha}\left(a - \frac{MGb}{c^2\alpha}\right) \overset{df}{=} \frac{2\tilde{a}}{\alpha} .$$

We can approximately solve for $u_<$ and $u_>$ by pretending $\alpha$ is a constant, then (if necessary) solve the resulting equations by iteration:

$$u_{>,<} = \frac{\tilde{a}}{\alpha} \pm \left[\left(\frac{\tilde{a}}{\alpha}\right)^2 - \frac{b}{\alpha}\right]^{1/2}.$$

To get the Newtonian limit we must take $\alpha=1$, giving

$$\tilde{a} = \frac{MGm^2}{J^2},$$

$$u_>^{(0)} = \tilde{a} + \sqrt{\tilde{a}^2 - b}$$

$$u_<^{(0)} = \tilde{a} - \sqrt{\tilde{a}^2 - b}\,.$$

defining $2\phi = MG\left(u_> + u_<\right)$ we have

$$u_> \approx \left(\tilde{a} + \sqrt{\tilde{a}^2 - b(1-4\phi)}\right)\left(1 + 4\phi\right) \approx u_>^{(0)}\left[1 + \frac{4\phi\, u_>^{(0)}}{u_>^{(0)} - u_<^{(0)}}\right]$$

and, interestingly, the same formula for $u_<$

$$u_< \approx \left(\tilde{a} - \sqrt{\tilde{a}^2 - b(1-4\phi)}\right)\left(1 + 4\phi\right) \approx u_<^{(0)}\left[1 + \frac{4\phi\, u_<^{(0)}}{u_>^{(0)} - u_<^{(0)}}\right]$$

In order that the two (inverse) radii be equal we must have

$$b\alpha = \tilde{a}^2,$$

$$u_> = u_< = \alpha\tilde{a} = \frac{\tilde{a}}{1 + 4MG\tilde{a}},$$

$$b = \alpha\tilde{a} = \tilde{a}^2\left(1 + 4MG\tilde{a}\right).$$

Obviously an orbit that satisfies the condition for circularity in the Newtonian approximation will no longer be circular when the general relativistic corrections are taken into account. Of course, even in general relativity circular orbits are possible.

2. A scalar particle's quantum-mechanical equation of motion in flat space is

$$\left[\partial^\mu \partial_\mu + \left(mc/\hbar\right)^2\right]\Phi = 0\,;$$

convert this to a general-relativistic equation of motion, taking the background metric to be the Schwarzschild metric of a massive static point source.

**Solution:**

The generalization of $\partial^\mu \partial_\mu \Phi$ is

$$\left(\Phi_{,\mu}\right)^{;\mu} = \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\; g^{\mu\nu}\,\Phi_{,\nu}\right)$$

which in the Schwarzschild metric is (spherical polar coordinates)

hence

$$\frac{1}{B(r)}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B(r)\frac{\partial\Phi}{\partial r}\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) - \frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\varphi^2} + \left(mc/\hbar\right)^2\Phi = 0\,,$$

where

$$B(r) = 1 - \frac{2MG}{r}\,.$$

3. From the solution of problem #2 above, take the nonrelativistic limit and identify the gravitational potential acting on the particle, through the first post-Newtonian correction.

**Solution:**

We write the solution in the form

$$\Phi(\vec{r}\,,t) = e^{-iE/\hbar}\,\Psi(\vec{r}\,)$$

and obtain

$$\frac{1}{\hbar^2c^2}\left[\frac{E^2}{B(r)} - m^2c^4\right]\Psi = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2B(r)\frac{\partial\Psi}{\partial r}\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) - \frac{1}{r^2\sin\theta}\frac{\partial^2\Psi}{\partial\varphi^2}$$

$$= -\nabla^2\Psi - \nabla B(r)\cdot\nabla\Psi + \frac{2MG}{rc^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)$$

Now in the nonrelativistic limit we have

$$E = mc^2 + \varepsilon$$

and we note that $\varepsilon$ and $MG/r$ are comparable in size, hence to lowest order,

$$\varepsilon\,\Psi \approx -\frac{1}{2m}\nabla^2\Psi - \frac{MmG}{r}\Psi\,;$$

to the next order, we must keep terms like

$$\frac{\varepsilon^2}{2mc^2}\,,\quad \frac{MG\varepsilon}{rc^2}\,,\quad \frac{MG}{r\,mc^2}\nabla^2\,,\quad etc.$$

The result is

$$\varepsilon\left(1 + \frac{\varepsilon}{2mc^2}\right)\Psi \approx -\frac{MG(m+\varepsilon/c^2)}{r}\Psi + \frac{MG}{mc^2}\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) - \frac{1}{2m}\nabla^2\Psi + \frac{MG}{r^2\,mc^2}\frac{\partial\Psi}{\partial r}$$

4. Again, from the solution of problem #2 above, use the eikonal approximation to calculate the gravitational deflection of a particle of large 3-momentum as it passes close to the Sun. How does this compare with the classical result? (The eikonal approximation for scattering can be found in the book *Collision Theory* by M.L. Goldberger and K.M. Watson.)

**Solution:**

Let $\Psi = e^{ipz + i\delta(b,z)}$ where $p$ is large. Since $m^2 = E^2 - p^2$ we have

We can neglect terms of order $ip$ to get

$$\frac{2MGE^2}{r} + p^2 \approx \left(p + \frac{\partial\delta}{\partial z}\right)^2 - \frac{2MG}{r}p^2\cos^2\theta$$

or

$$\frac{\partial\delta}{\partial z} \approx -\frac{MG}{rp}\left(E^2 + p^2\cos^2\theta\right).$$

Since $\cos\theta = \frac{z}{r}$ and since the phase may be identified with the classical potential *via*

$$\frac{\partial\delta}{\partial z} \approx -\frac{E}{p}V_{class}\ ,$$

we can calculate the transverse force,

$$F_\perp = -\frac{\partial V_{class}}{\partial x} = -\frac{MGE\,x}{r^3\,c^2}\left(1 + 3\frac{p^2}{E^2}\frac{z^2}{r^2}\right).$$

If we integrate this over all time (with $dt = dz/v$ ) we obtain the nonrelativistic limit for scattering from a Coulombic potential ($p << E$) or the Einstein value for high velocity, using

$$\int_{-\infty}^{+\infty}\frac{d\zeta}{\left(1+\zeta^2\right)^{3/2}} = 2\,,$$

$$\int_{-\infty}^{+\infty}\frac{d\zeta\,\zeta^2}{\left(1+\zeta^2\right)^{5/2}} = \frac{2}{3}\,;$$

that is,

$$\Delta\theta \approx \frac{1}{p}\int_{-\infty}^{+\infty}dt\,F_\perp = -\frac{2MG}{b\,c^2}\frac{E}{pv}\left(1 + \frac{p^2\,c^2}{E^2}\right).$$

The same result can be obtained by evaluating the (approximate) scattering amplitude, $f(\theta)$, and calculating the expected scattering angle

$$\langle\theta\rangle = \frac{\int_0^\pi d\theta\,|f(\theta)|^2\,\theta}{\int_0^\pi d\theta\,|f(\theta)|^2}\,;$$

we leave this as a further exercise for the student. ☺