HW #8—Solutions

1. A white dwarf star has the solar mass and a radius ≈ 9000 Km . What is the average Fermi momentum of the electrons (in MeV/c), assuming the star is made of fully ionized helium? Are the electrons relativistic or nonrelativistic?

Solution:

The mass is 2×10^{33} gm, or about 3×10^{56} He nuclei; thus the electron number Z is thus 6×10^{56}; the radius is 9×10^{8} cm, hence the electron number density is

$$n_e \approx 2\times10^{29}\ /\mathrm{cm}^3 = \frac{p_F^3}{3\pi^2\,(\hbar c)^3}$$

or $p_F \approx 0.35$ MeV/c . The electrons have velocity

$$v/c = \frac{p_F}{\sqrt{p_F^2 + m_e^2}} = 0.57$$

hence the electrons are relativistic.

2. Estimate the tidal stress acting on a rod of cross-section A and length L falling lengthwise through the event-horizon of a black hole of mass $M = 2\times10^{33}$ gm . Assume the rod is steel: what is the Roche's limit for the rod (smallest distance it can approach the black hole)?

Solution:

We need the acceleration of points on the rod, as measured in the (freely falling) instantaneous rest frame of the center of the rod, relative to the center of the rod. Since $\frac{d^2 x^\mu}{d\tau^2}$ is a vector, and since

$$(d\tau)^2 = (dt')^2 - (dr')^2 = B(r)\,(dt)^2 - B(r)^{-1}\,(dr)^2\ ,$$

$$\frac{d^2 r'}{dt'^{\,2}} \equiv \frac{d^2 r'}{d\tau^2} = \frac{d^2 r}{d\tau^2}\frac{\partial r'}{\partial r} + \frac{d^2 t}{d\tau^2}\frac{\partial t'}{\partial t} = \frac{1}{B(r)}\frac{d^2 r}{d\tau^2} - B(r)\frac{d^2 t}{d\tau^2}$$

We use the orbital equations with zero azimuthal motion, corresponding to free fall from $r = \infty$, with initial velocity 0:

$$\frac{d}{d\tau}\left(B(r)\frac{dt}{d\tau}\right) = 0\ ,$$

$$\frac{dr}{d\tau} = -\left[1 - B(r)\left(1 - \frac{2E}{m}\right)\right]^{1/2} \equiv -\sqrt{\frac{2MG}{r}}\ .$$

These give the acceleration of any point relative to the center, as measured in the instantaneous rest frame, as (note that to an observer at the center of the rod, the acceleration is 0)

$$\frac{d^2 r'}{dt'^2} - \left.\frac{d^2 r'}{dt'^2}\right|_{center} = -\frac{1}{B(r)}\frac{dB}{dr}\left[\frac{1}{2} + \frac{dr}{d\tau}\right].$$

The relative force on any point with mass $\delta m = \rho A\, d\xi$ located a distance $\delta r = B(r)\xi$ from the center is

$$df_r = \frac{MG}{r^3}\left[\frac{-1}{B(r)}\left(\frac{1}{2} - \sqrt{\frac{2MG}{r}}\right)\left(\frac{2MG}{r} - 4\right) + \sqrt{\frac{2MG}{r}}\right]\rho A\xi\, d\xi = \Gamma(r)\,\rho\, A\xi\, d\xi$$

Thus we have net stress, from one end to the other,

$$T = \frac{1}{A}\int_{-L/2}^{+L/2} |df_r| = \Gamma(r)\,\rho\,\frac{L^2}{2}.$$

The density of steel is ≈ 5 gm/cm and its tensile strength is 3×10^{10} dyne/cm^2, hence for a 1 meter long rod,

$$r_{Roche} \approx \frac{R_S}{0.027} = 110\ \text{Km} >> R_S$$

(Note that to get a dimensionally correct expression, factors of c must be supplied.)

3. Show that the trajectory of light rays in the Schwarzschild metric obeys

$$\frac{d^2u}{d\varphi^2} + u = 3u^2$$

where $u = \frac{MG}{r}$, r is the radial coordinate and φ is the azimuthal angle.

Solution:

Clearly, for light $d\tau = 0$, hence, in the equatorial plane,

$$B(r)\left(\frac{dt}{d\varphi}\right)^2 - \frac{1}{B(r)}\left(\frac{dr}{d\varphi}\right)^2 - r^2 = 0$$

Moreover,

$$B(r)\frac{dt}{d\varphi} = \Gamma = \text{const.} \qquad \text{(from Euler-Lagrange equ'n).}$$

Thus, with $r = \frac{MG}{u}$,

$$\frac{dr}{d\varphi} = -\frac{MG}{u^2}\frac{du}{d\varphi},$$

hence

$$\left(\frac{MG}{u^2}\right)^2\left(\frac{du}{d\varphi}\right)^2 = \Gamma^2 - \left(\frac{MG}{u}\right)^2(1 - 2u).$$

Multiply by u^2 and differentiate once w.r.t. φ :

$$2\,\frac{du}{d\varphi}\,\frac{d^2u}{d\varphi^2} = \left(6u^2 - 2u\right)\frac{du}{d\varphi}, \qquad \text{QED.}$$

4. Paul is orbiting a neutron star at a distance $4MG/c^2$, in a circular orbit. Peter, his twin brother, is fired radially outward from the surface with less than escape velocity. His path crosses Paul's orbit just as Paul comes by, so they synchronize their clocks. On his way back down, Peter again encounters Paul as their trajectories cross, Paul having completed 10 orbits between their meetings. They again compare clocks. How much do their clocks disagree?

Solution:

The meetings of the two brothers must occur at common values of space-time coordinates. Thus we know that they meet at

$t = t_1 = 0$
$r = R$
$\varphi = 0$
$\theta = \pi/2$

and again at

$t = t_2$
$r = R$
$\varphi = 20\pi$
$\theta = \pi/2$.

For Paul's circular orbit we may choose $B(R)\,\dfrac{dt}{d\tau} = 1$, and of course $\dfrac{dr}{d\tau} = 0$, so that

$$B(R)\left(\frac{dt}{d\tau}\right)^2 - R^2\left(\frac{d\varphi}{d\tau}\right)^2 = 1\ ;$$

moreover, $B(R) = 1/2$, so that

$$\left(\frac{d\varphi}{d\tau}\right) = \frac{1}{R}$$

or

$$\Delta\tau_{Paul} = 80\pi\,\frac{MG}{c^3}$$

$$t_2 = \frac{1}{B(R)}\,\Delta\tau_{Paul} = 160\pi\,\frac{MG}{c^3}\ .$$

Now let us examine Peter's trajectory:

$$B(r)\,\frac{dt}{d\tau} = \Gamma = \text{const.}$$

and since $\frac{d\varphi}{d\tau} = 0$,

$$B(r)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{B(r)}\left(\frac{dr}{d\tau}\right)^2 = 1$$

or

$$\Gamma^2 - \left(\frac{dr}{d\tau}\right)^2 = B(r) .$$

Clearly, $\Gamma^2 = B(r_>)$, where $r_>$ is the apstellion (??). We also see that

$$\frac{dt}{dr} \equiv \frac{dt}{d\tau}\left(\frac{dr}{d\tau}\right)^{-1} = \frac{\Gamma}{B(r)}\left(\Gamma^2 - B(r)\right)^{-1/2}$$

or

$$t_2 = 2\int_R^{r_>} dr\, \frac{\Gamma}{B(r)}\left(\Gamma^2 - B(r)\right)^{-1/2} = 160\pi\frac{MG}{c^3} .$$

Peter's elapsed proper time is

$$\Delta\tau_{Peter} = 2\int_R^{r_>} dr\left(\Gamma^2 - B(r)\right)^{-1/2} .$$

Define a new variable $u = \frac{2MG}{r}$; then

$$\int_{u_<}^{1/2} \frac{du}{u^2(1-u)}\left(\frac{1-u_<}{u-u_<}\right)^{1/2} = 40 .$$

But the factor

$$\frac{\Gamma}{B(r)} = \frac{\sqrt{1-u_<}}{1-u}$$

is never much larger than 1 (since the integrand is dominated by small u near $u_<$) hence

$$t_2 = 160\pi\frac{MG}{c^3} \approx \Delta\tau_{Peter}$$

i.e. $\Delta\tau_{Peter} \approx 2\,\Delta\tau_{Paul}$.

5. A two-dimensional surface can be represented in 3-dimensional Cartesian coordinates as

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 ,$$

where a, b and c are constants.

a) Find a suitable two-dimensional (parametric) representation of the surface.
b) Find an expression for the (differential) distance between two points *in the surface*, thereby constructing the metric tensor.
c) Find the *shortest* distance, in the surface, between the points $z = +c$ and $x = +a$.
d) Construct the curvature tensor, the Ricci tensor and the curvature scalar.

e) What is the area of the surface? (Express it as an integral, it can't be done in closed form.) Check limiting cases to verify your result.

Hints: Examine the spherical case to see what to do about part a). In part b) the answer is *not* $(dx)^2+(dy)^2+(dz)^2$. Use vectorial methods to restrict displacements to the surface.

Solution:

My answer to part a) is

$$x = a\sin\theta\cos\varphi$$

$$y = b\sin\theta\sin\varphi$$

$$z = c\cos\theta \ .$$

Next we see that (non-orthogonal) direction differentials in the surface are defined by

$$d\vec{u} = \frac{\partial\vec{r}}{\partial\theta}\,d\theta$$

$$d\vec{v} = \frac{\partial\vec{r}}{\partial\varphi}\,d\varphi \ .$$

Manifestly, the distance (in the surface!) between nearby points is

$$\begin{aligned}(ds)^2 &= \left(d\vec{u}\right)^2 + \left(d\vec{v}\right)^2 \\ &= \left(a^2\cos^2\theta\cos^2\varphi + b^2\cos^2\theta\sin^2\varphi + c^2\sin^2\theta\right)(d\theta)^2 + \\ &\qquad + \frac{1}{2}\left(b^2 - a^2\right)\sin2\theta\ \sin2\varphi\ d\theta\,d\varphi + \\ &\qquad + \left(a^2\sin^2\varphi + b^2\cos^2\varphi\right)\sin^2\theta\ (d\varphi)^2\end{aligned}$$

—note this reduces to the usual result for a sphere!

Therefore,

$$g_{\theta\theta} = a^2\cos^2\theta\cos^2\varphi + b^2\cos^2\theta\sin^2\varphi + c^2\sin^2\theta$$

$$g_{\theta\varphi} = \frac{1}{4}\left(b^2 - a^2\right)\sin2\theta\ \sin2\varphi$$

$$g_{\varphi\varphi} = \left(a^2\sin^2\varphi + b^2\cos^2\varphi\right)\sin^2\theta \ .$$

We note that in our parameterization, $z=c$ is the North pole, $\theta=0$, and x=a is represented by $(\theta=\pi/2\,,\ \varphi=0)$. Thus the easiest way to find the shortest distance between these two points is

$$\delta\int_0^{\pi/2} d\theta\left[g_{\theta\theta} + 2g_{\theta\varphi}\frac{d\varphi}{d\theta} + g_{\varphi\varphi}\left(\frac{d\varphi}{d\theta}\right)^2\right]^{1/2} = 0\ .$$

This gives an Euler-Lagrange equation of the form

$$\frac{d}{d\theta}\left(\frac{g_{\theta\varphi} + g_{\varphi\varphi}\frac{d\varphi}{d\theta}}{\sqrt{etc.}}\right) = \frac{\partial}{\partial\varphi}\sqrt{etc.}$$

which must be solved numerically, except in the special case $a=b$, for which $g_{\theta\varphi} = 0$ and there is no φ–dependence:

$$\left(g_{\varphi\varphi}\frac{d\varphi}{d\theta}\right)^2 = A^2\left[g_{\theta\theta} + g_{\varphi\varphi}\left(\frac{d\varphi}{d\theta}\right)^2\right]$$

or

$$\left(\frac{d\varphi}{d\theta}\right)^2 = \frac{A^2 g_{\theta\theta}}{g_{\varphi\varphi}^2 - g_{\varphi\varphi}A^2}.$$

Manifestly the constant A must vanish or we could find $\left(\frac{d\varphi}{d\theta}\right)^2 < 0$ near $\theta=0$. Thus

$$d_{\min} = \int_0^{\pi/2} d\theta \left(a^2\cos^2\theta + c^2\sin^2\theta\right)^{1/2}$$

which is a standard elliptic integral.

In terms of the directional differentials $d\vec{u}$, $d\vec{v}$ the element of area is

$$d\vec{A} = d\vec{u}\times d\vec{v}.$$

Since we don't care about the direction,

$$A = \iint_S |d\vec{u}\times d\vec{v}| = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sin\theta\left[(ab)^2\cos^2\theta + c^2\sin^2\theta\left(b_2\cos^2\varphi + a^2\sin^2\varphi\right)\right]^{1/2}.$$

This can be reduced to the form

$$A = \int_0^{2\pi} d\varphi\left(ab + \frac{\Delta\sin^{-1}\left(\sqrt{1-(ab/\Delta)^2}\right)}{\sqrt{1-(ab/\Delta)^2}}\right)$$

where

$$\Delta^2 = c^2\left(b^2\cos^2\varphi + a^2\sin^2\varphi\right).$$

The integral cannot be further simplified unless $b=a$, in which case $\Delta = ac$ and the integrand becomes independent of φ. The result is then

$$A = 2\pi a\left(a + \frac{c\sin^{-1}\left(\sqrt{1-(a/c)^2}\right)}{\sqrt{1-(a/c)^2}}\right).$$

We see this reduces to the proper result, $4\pi a^2$ in the limit $c \to a$.

Finally we construct the curvature tensor, Ricci tensor and curvature scalar. The inverse of the metric tensor is

$$g^{\theta\theta} = \frac{1}{g}g_{\varphi\varphi}, \qquad g^{\varphi\varphi} = \frac{1}{g}g_{\theta\theta}, \qquad g^{\theta\varphi} = \frac{-1}{g}g_{\theta\varphi}$$

There are six (6) Christoffel symbols, and only one (1) non-zero component of the curvature tensor.

We do the special case with $a=b$. The φ–dependence vanishes and the metric becomes diagonal. Then

$$R_{\theta\varphi\,\theta\varphi} = \frac{1}{2}\partial_\theta^2 g_{\varphi\varphi} - \frac{1}{4}\left[g^{\theta\theta} g_{\theta\theta\,,\,\theta} g_{\varphi\varphi\,,\,\theta} + g^{\varphi\varphi}\left(g_{\varphi\varphi\,,\,\theta}\right)^2 \right]$$

$$R_{\varphi\varphi} = g^{\theta\theta} R_{\theta\varphi\,\theta\varphi}$$

$$R_{\theta\theta} = g^{\varphi\varphi} R_{\theta\varphi\,\theta\varphi}$$

$$R_{\theta\varphi} = 0$$

$$R = 2 g^{\varphi\varphi} g^{\theta\theta} R_{\theta\varphi\,\theta\varphi}$$

The derivatives are

$$g_{\varphi\varphi\,,\,\theta} = 2a^2 \sin\theta\cos\theta$$

$$g_{\theta\theta\,,\,\theta} = 2\left(c^2 - a^2\right)\sin\theta\cos\theta$$

$$\frac{\partial^2}{\partial\theta^2} g_{\varphi\varphi} = 2a^2\cos 2\theta$$

so the computation of the curvature scalar and the tensors just becomes a moderately tedious chore.