HW #9

1. The Bargmann-Wigner inequality says the number of bound states of given orbital angular momentum l, in a spherically symmetric potential $V(r)$, is bounded above by

$$n_l \leq \frac{2m}{\hbar^2}\frac{1}{2l+1}\int_0^\infty dr\, r\, |V(r)|$$

Manifestly, if $n_0 < 1$ there are no bound states even if the potential is attractive.

As you may know, the Coulomb potential in a degenerate electron gas is effectively screened by the rearrangement of the other charges, so the potential between an electron and a nucleus is roughly

$$V(r) \frac{-Ze^2}{r} e^{-\lambda r}$$

where the (Debye) screening constant is

$$\lambda^2 = \frac{4\varepsilon_F k_F e^2}{\pi}.$$

Use this fact to explain why all atoms will eventually be ionized in a sufficiently dense star.

2. The equation of hydrostatic equilibrium for a (degenerate) electron gas mixed with ^{4}He (as in a white dwarf star) is

$$4\pi r^2\, dp = -r^{-2}\, GM(r)\, dM(r) \qquad (1)$$

where

$$dM(r) = 4\pi r^2\, \rho(r)\, dr \qquad (2)$$

and the mass-density $\rho(r)$ arises from the ^{4}He nuclei which comprise the major fraction of the stellar mass. Clearly, $\rho(r) \approx 4 m_N B(r)$ where m_N is the nucleon mass and $B(r)$ is the baryon number density.

The pressure is given by

$$p(r) = \frac{8\pi}{3\hbar^3 c}\int_0^{k_F(r)} dk\, k^4 \left(k^2 + m_e^2 c^2\right)^{-1/2} \qquad (3)$$

where

$$k_F(r) = \hbar\left(\frac{3\pi^2 \rho(r)}{m_N \mu}\right)^{1/3} \qquad (4)$$

and μ is the number of nucleons/electron, $\mu \approx 2$ in dwarf stars.

Combine Eq. 3 and Eq. 4 with Eq. 1 and Eq. 2 and show that no solution can exist if the electron mass is neglected. A physically acceptable solution has

$M(0) = 0$, $\quad p(R) = 0$

$M(R) = M$, $\quad k_F(r) = 0$.

3. Integrate the above equations numerically (write a simple computer program!), taking $k_F(r=0)$ as a parameter, for non-zero electron mass, and find the upper limit on the mass of a white dwarf star.

4. Consider a star of mass $20M_{sol}$ that has reached the end of thermonuclear burning. That is, imagine it has been in hydrostatic equilibrium when the power is suddenly shut off. The innermost core, of mass—say—$2M_{sol}$, suddenly shrinks to the radius of a neutron star.

   a) How much energy is released in this shrinkage?

   b) Assume the outer $18M_{sol}$ hasn't time to fall, and can be ejected. What is the energy needed to eject it (that is, to remove it to $r = \infty$)?

Hint: The radius and density distribution of an arbitrary star scales approximately as the mass, *i.e.* homologously, as long as nuclear energy generation is taking place in the core.