Solutions to HW # 9

1. The Bargmann–Wigner inequality gives

$$n_{l=0} \le \frac{2m}{\hbar^2}\int_0^\infty dr\, r\,\frac{Ze^2}{r}\,e^{-\lambda r} = \frac{2mc}{\hbar}\,\frac{Z\alpha}{\lambda} = Z\left(\lambda\!\!\!^{-3}_C\, n_e\right)^{-1/2}\sqrt{\frac{2\alpha}{3\pi}}$$

where n_e is the electron density, $\alpha = \dfrac{e^2}{\hbar c} \approx \dfrac{1}{137}$ is the fine structure constant, and $\lambda\!\!\!^{-}_C$ is the Compton wavelength of the electron, $\lambda\!\!\!^{-}_C = \dfrac{\hbar}{mc} \approx 4\times10^{-11}$ cm . Thus the number of bound states falls to <1 at sufficiently high density.

2. If the electron mass is neglected, the equation of state becomes that of a polytrope with

$$p \propto \rho^{4/3}$$

i.e. the equation of static equilibrium becomes

$$\frac{K}{\pi G}\frac{d}{dr}\left(r^2\,\frac{d}{dr}\,\rho^{1/3}\right) = -r^2\,\rho\ .$$

This means that if we rescale $r \to Rx$ and $\rho \to \rho_c\, y$, there is an overall dimensionless parameter $\left(R^3\,\rho_c\right)^{2/3}$. That is, the problem can yield no definite radius because there is no way to determine this parameter (such as an eigenvalue condition). Put another way, if the density vanishes at the radius of the star, then because it is monotonically decreasing, it must be infinite at the origin.

3. The equations to be solved are

$$r^2\,dp = -GM(r)\,\rho(r)\,dr$$

$$dM(r) = 4\pi\,r^2\,\rho(r)\,dr$$

$$p(r) = \frac{8\pi c}{3h^3}\int_0^{k_F(r)} dk\,k^4\left(k^2 + m_e^2c^2\right)^{-1/2}$$

$$k_F(r) = \hbar\left(\frac{3\pi^2\,\rho(r)}{m_N\,\mu}\right)^{1/3}\ .$$

We rescale as follows:

$$r \to Rx$$

$$k_F(Rx) \to m_e c\, y(x)$$

$$M(r) \rightarrow M_c\, B(x)$$

then with the change of variable $A(x) = \sqrt{y^2(x) + 1}$ we find

$$dA = -\left[\frac{GM_c}{Rc^2}\,\frac{2M_N}{m_e}\right]\frac{B(x)}{x^2}\,dx$$

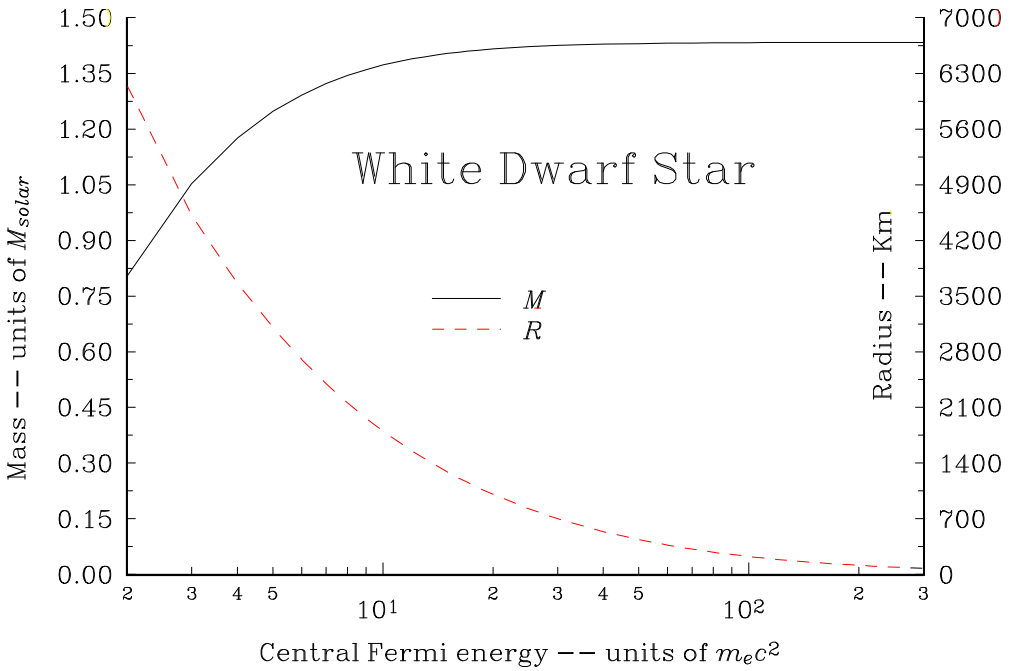
$$dB = \left[\frac{4}{3\pi}\,\frac{2m_N}{M_c}\left(\frac{m_e c\, R}{\hbar}\right)^3\right]\left(A^2 - 1\right)^{3/2} x^2\,dx$$

With appropriate choices of the scaling constants M_c and R we can make both the bracketed constants unity, giving

$$dA = -\frac{B(x)}{x^2}\,dx$$

$$dB = \left(A^2 - 1\right)^{3/2} x^2\,dx\,.$$

The differential equations can be solved by, *e.g.* Runge-Kutta or any other technique, taking an initial value for A (at $x=0$) larger than 1 (since it is the ratio of the electron energy to its rest mass). The solution is plotted below, and the program that produced it is given on the next page. (I will also post a BASIC equivalent for those who don't speak Forth.)



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\ Numerical solution of the Chandrasekhar equations of a
\ white dwarf star
\
marker -STAR
needs ftran108.f

\ data structures
f# 0.71    FCONSTANT S.MASS     \ mass scale in Solar mass units
f# 3853    FCONSTANT S.RADIUS   \ radius scale in Km
fvariable A     \ local Fermi energy in terms of electron mass
fvariable dA
fvariable dAp
fvariable B     \ stellar mass in units of 2Z x proton mass
fvariable dB
fvariable dBp
fvariable x     \ radius in dimensionless units
fvariable dx    \ radial step-size

\ Runge-Kutta algorithm
\   y' = f(x,y)
\   K = dx * f(x,y)      L = dx * f(x+dx, y+K)
\   y = y + (K + L) / 2

: ?dBp      ( f -)     \ calculate dBp according to condition flag
    IF    f" sqrt( (A+dA)^2 - 1 ) * ( (A+dA)^2 - 1 ) * x^2 * dx "
    ELSE  f0.0
    THEN  dBp  f!  ;

: x_step        \ advance solution by 1 step in both equations
    f" dB = x^2 * dx * (A^2 - 1) * sqrt(A^2 - 1)"
    f" dA = -dx * B / x^2 "
    f" x = x + dx"
    f" A+dA -1.0 "  f0>   ?dBp
    f" dAp =  -dx * (B+dB) / x^2 "
    f" A = A + 0.5 * ( dA + dAp )"
    f" B = B + 0.5 * ( dB + dBp )"  ;

: RESULTS       \ display output
    B f@  S.MASS  F*  fe.   x f@  S.RADIUS  F*  fe.  ;

: INTEGRATE    ( f: A[0] dA - )
    dx f!  A f!   f" x = dx"            \ initialize
    f0.0   B f!
        BEGIN  A f@  f1.0 F>
        WHILE  x_step
        REPEAT  RESULTS  ;

\ say  f# central_fermi_energy  f# radial increment  INTEGRATE
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4. Consider a star of mass 20 $M_\odot$ that has reached the end of thermonuclear burning. That is, imagine it was in hydrostatic equilibrium when the power is suddenly shut off. The innermost core, of mass—say—2 $M_\odot$, suddenly shrinks to the radius of a neutron star.

a) how much energy is released in this shrinkage?

Solution

The radius of a star of 2 $M_\odot$ in hydrostatic equilibrium is roughly 2 $R_\odot$ under the homologous assumption, i.e. about 1.4×10^6 Km. Thus the energy release is on dimensional grounds

$$\Delta E = \text{constant} \times \left(\frac{M^2 G}{R_{NS}} - \frac{M^2 G}{R_0} \right) \approx \text{constant} \times \frac{M^2 G}{R_{NS}}$$

The constant must determined from the binding energy of a neutron star, but we replace it by unity for simplicity, giving ($R_{NS} \approx 10$ Km)

$$\Delta E \approx 10^{54} \text{ ergs.}$$

b) Assume 18 $M_\odot$ hasn't time to fall and is ejected. What is the energy needed to eject it (that is, to remove it to $r = \infty$ with zero kinetic energy)?

Solution

Since the radii are large and the dimensionless gravitational potentials $\frac{2GM}{Rc^2}$ are small, we may use Newtonian gravity to estimate the gravitational binding energy. The formula for this is

$$V_{grav} = -\frac{1}{2} G \int d^3r\, \rho(r) \int d^3r'\, \rho(r') \frac{1}{|\vec{r} - \vec{r}'|}$$

where

$$4\pi \rho(r) \approx \frac{M_\odot}{R^2 R_\odot}.$$

Then

$$V_{grav} \approx -\frac{GM_\odot^2}{R_\odot} \int_2^{20} dx \frac{1}{x} \int_x^{20} dx' = -\frac{GM_\odot^2}{R_\odot} \left[20 \ln\left(20/2\right) - (20 - 2) \right]$$

$$\approx -28 \frac{GM_\odot^2}{R_\odot} = 1.07\times10^{50} \text{ ergs}.$$