HW # 10

1. Consider a 3-dimensional space of constant curvature, whose metric is given by

$$(ds)^2 = d\vec{r}\cdot d\vec{r} + \frac{(d\vec{r}\cdot\vec{r})^2}{a^2 - \vec{r}\cdot\vec{r}}$$

a) What is the curvature of the space?

b) What is the shape of the space for $a^2 > 0$ as well as $a^2 < 0$?

c) Assuming $a^2 > 0$, what is the radius and the circumference of the circle of constant $r = b$, whose center is the origin, $r = 0$?

Hint: Use polar coordinates,

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta \ .$$

Solution:

Let r, θ , φ be the coordinates. Then

$$(ds)^2 = \frac{(dr)^2}{1 - r^2/a^2} + r^2\,(d\theta)^2 + r^2\sin^2\theta\ (d\varphi)^2 \ .$$

The metric tensor is diagonal, with components

$$g_{rr} = \frac{1}{1 - r^2/a^2}$$

$$g_{\theta\theta} = r^2$$

$$g_{\varphi\varphi} = r^2\sin^2\theta \ .$$

The curvature is

$$R = g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi}$$

where

$$R_{\beta\nu} = \left\{ \begin{matrix} \alpha \\ \beta\ \nu \end{matrix} \right\}_{,\,\alpha} + \left\{ \begin{matrix} \alpha \\ \beta\ \nu \end{matrix} \right\} \frac{1}{\sqrt{g}}\,\partial_\alpha\sqrt{g} - \left\{ \begin{matrix} \sigma \\ \beta\ \alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \sigma\ \nu \end{matrix} \right\} - \partial_\nu\left(\frac{1}{\sqrt{g}}\,\partial_\beta\sqrt{g}\right).$$

Working out the 18 Christoffel symbols, differentiating and reassembling gives

$$R = \frac{6}{a^2} \ .$$

When $a^2 > 0$, a is the maximum possible value of r, hence the space is closed. With $a^2 < 0$ there is no maximum value of r, hence the space is unbounded. In either case the space is spherically symmetric with constant curvature.

Finally, the radius of the circle with $r = b$ is

$$\rho = \int_0^b dr \left(\frac{1}{1 - r^2/a^2} \right)^{1/2} = a \sin^{-1}\left(\frac{b}{a}\right)$$

and the circumference is just $2\pi b$.

2. Define 2 vectors in Minkowski space

$$A_\mu = \left(0, -1, 0, 0\right)$$

and

$$B_\mu = \left(0, 0, -1, 0\right).$$

From these, define the two polarization tensors

$$\varepsilon^{(1)}_{\mu\nu} = A_\mu B_\nu + A_\nu B_\mu$$

and

$$\varepsilon^{(2)}_{\mu\nu} = A_\mu A_\nu - B_\nu B_\mu .$$

Show that under rotations through angle φ about the z-axis the two tensors transform as

$$\varepsilon^{(1)}_{\mu\nu} \to \varepsilon^{(1)}_{\mu\nu} \cos(2\varphi) - \varepsilon^{(2)}_{\mu\nu} \sin(2\varphi)$$

$$\varepsilon^{(2)}_{\mu\nu} \to \varepsilon^{(1)}_{\mu\nu} \sin(2\varphi) + \varepsilon^{(2)}_{\mu\nu} \cos(2\varphi)$$

What do you deduce from this result, about the spin of the gravitational quantum?

Solution:

Rotating about the z-axis, we get

$$A_\mu \to A_\mu' \cos\varphi - B_\mu' \sin\varphi$$

$$B_\mu \to A_\mu' \sin\varphi + B_\mu' \cos\varphi$$

hence

$$\begin{aligned}
\varepsilon^{(1)}_{\mu\nu} &= \left(A_\mu' \cos\varphi - B_\mu' \sin\varphi\right)\left(A_\nu' \sin\varphi + B_\nu' \cos\varphi\right) + \\
&+ \left(A_\nu' \cos\varphi - B_\nu' \sin\varphi\right)\left(A_\mu' \sin\varphi + B_\mu' \cos\varphi\right) \\
&= \left(A_\mu' B_\nu' + A_\nu' B_\mu'\right)\left(\cos^2\varphi - \sin^2\varphi\right) + \left(A_\mu' A_\nu' - B_\mu' B_\nu'\right) 2 \sin\varphi \cos\varphi \\
&= \varepsilon^{(1)}_{\mu\nu}{}' \cos(2\varphi) + \varepsilon^{(2)}_{\mu\nu}{}' \sin(2\varphi) .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varepsilon^{(2)}_{\mu\nu} &= \left(A_\mu' \cos\varphi - B_\mu' \sin\varphi\right)\left(A_\nu' \cos\varphi - B_\nu' \sin\varphi\right) - \\
&- \left(A_\mu' \sin\varphi + B_\mu' \cos\varphi\right)\left(A_\nu' \sin\varphi + B_\nu' \cos\varphi\right) \\
&= - \varepsilon^{(1)}_{\mu\nu}{}' \sin(2\varphi) + \varepsilon^{(2)}_{\mu\nu}{}' \cos(2\varphi) .
\end{aligned}$$

Thus we conclude that, by analogy with the photon, which is a quantum of the vector field whose polarization vectors transform like

$$A_\mu \to A_\mu' \cos\varphi - B_\mu' \sin\varphi$$

$$B_\mu \to A_\mu' \sin\varphi + B_\mu' \cos\varphi$$

and therefore have intrinsic angular momentum $1\hbar$, because the angular dependence for the polarization tensors of a tensor radiation field is $\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}(2\varphi)$, we conclude the angular momentum of the graviton is $2\hbar$.

3. Solve the Oppenheimer-Volkov equation 25.16 (using the expressions 25.17a,b for energy density and pressure) numerically, in the same manner as you solved the Chandrasekhar equation. Find the maximum mass.

Solution

The equations of static equilibrium are

$$-r^2\frac{dp}{dr} = GM(r)\,\rho(r)\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{M(r)}\right)\left(1-\frac{2GM(r)}{r}\right)^{-1}$$

$$\frac{dM(r)}{dr} = 4\pi r^2\,\rho(r)\,dr$$

We need to introduce dimensional scaling parameters in order to reduce these equations to dimensionless form:

$$r \to Rx$$

$$M(r) \to M\,b(r)$$

$$p \to \frac{1}{8\pi^2}mc^2\left(\frac{mc}{\hbar}\right)^3\tilde{p}$$

$$\rho \to \frac{1}{8\pi^2}m\left(\frac{mc}{\hbar}\right)^3\tilde{\rho}$$

$$\varepsilon_F \to mc^2\,a$$

With these definitions, we find

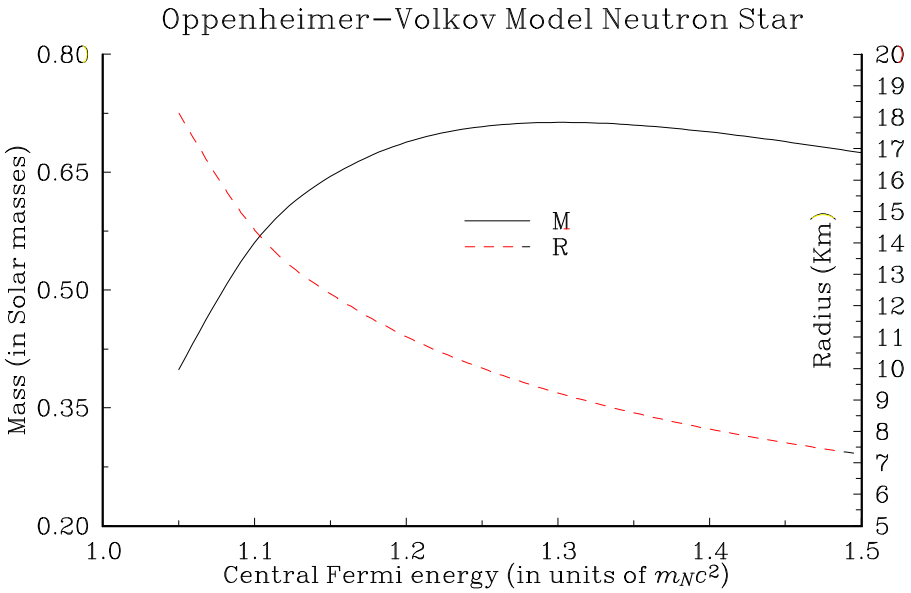
$$\frac{db}{dx} = \Lambda\tilde{\rho}x^2$$

and

$$-x^2\frac{da}{dx} = \frac{\lambda}{2}a\left(b+\Lambda\tilde{p}x^3\right)\left(1-\frac{\lambda b}{x}\right)^{-1}$$

where $\lambda = \frac{2MG}{Rc^2}$ and $\Lambda = \frac{m}{2\pi M}\left(\frac{Rmc}{\hbar}\right)^3$, m is the neutron mass, and p and ρ are given by the integrals 25.17a,b . Setting λ and Λ (note they are dimensionless) to 1 by appropriate choices of

M and R gives the equations solved in the appended (Forth) program, with the results graphed and tabulated below. Note there is a maximum possible mass, about 0.7 $M_{\odot}$.



Central Fermi energy	**Mass**	**Radius**
(in $m_N c^2$)	(in $M_{\odot}$)	(Km)
1.05	398.507E-3	18.1200E0
1.10	559.989E-3	14.4018E0
1.15	644.361E-3	12.3777E0
1.20	687.916E-3	11.0137E0
1.25	707.768E-3	10.0089E0
1.29	713.008E-3	9.36822E0
1.30	713.258E-3	9.22260E0
1.31	713.159E-3	9.08669E0
1.33	712.040E-3	8.82943E0
1.35	709.870E-3	8.59158E0
1.40	700.975E-3	8.07220E0
1.45	688.711E-3	7.64020E0
1.50	674.462E-3	7.26644E0

Results obtained with dx = 0.001

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\ Neutron star model: free-nucleon equation of state
\ Version of  April 20th, 1998
\ Runge-Kutta algorithm — see, e.g., Abramowitz & Stegun, p. 896
\   y' = f(x,y)
\   K = dx * f(x,y)     L = dx * f(x+dx, y+K)
\   y = y + (K + L) / 2

marker -NSTAR
needs ftran108.f

\ data structures
: scalars  0 DO  FVARIABLE  LOOP  ;

10 scalars   a b a1 b1 da db x dx p rho    \ define fp variables
    \ a: local (relativistic) Fermi energy / neutron mass
    \ b: dimensionless mass
    \ x: dimensionless radius
    \ p: dimensionless pressure— [a*sqrt(a^2-1)*(2*a^2-5)+3*acosh(a)]/3
    \ rho: dimensionless density— a*sqrt(a^2-1)*(2*a^2-1)-acosh(a)
    \       — NOTE: each must be divided by 8*pi^2 to get actual fn's.
    \ also: rho + p = (8/3)*[a^2-1]^(3/2)

\ Problem: to solve the coupled equations
 \   da = -(3/16)*dx*b*[rho + p]*[1 + p*x^3/b]/[1 - b/x] / x^2 /(a^2-1)^(3/2)
 \   db = rho * x^2 * dx
 \   the scaled-out factors are:
 \       MASS = (Mplanck)^3 * sqrt(pi/4) / m^2       m = neutron mass
 \       R    = (hbar/m/c) * (2*pi*MASS/m)^(1/3)

f# 1.645   FCONSTANT MASS/M0    \ mass scale in solar masses
f# 4.854   FCONSTANT R          \ radius scale in km

: p_and_rho                     \ calculate p and rho
     F" p = a * SQRT(a^2-1)"
     F" ACOSH(a)"              ( f: — acosh[a])
     F" p * (2*a^2 - 1)"   FOVER  F-  rho  F!
     F" p * (2*a^2 - 5)/3"   F+       p    F! ;

: da_and db
      F" da = -0.5* a * ( b/x^2 + p*x ) * dx / ( 1 - b/x )"
      F" db = x^2 * rho * dx"  ;

: x_step    \ Runge-Kutta O(h^2) integration
      p_and_rho   da_and db     \ compute steps

      F" a1 = a"  F" b1 = b"    \ save values @ x

      F" a = a + da"            \ guess values @ x + dx
      F" b = b + db"

      F" x = x + dx"            \ inc. x

      p_and_rho   da_and db     \ recompute steps

      F" a = (a + da + a1)/2"   \ add up
      F" b = (b + db + b1)/2"   ;
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: INITIALIZE   ( f: a[0] dx —  )
       dx F!  a F!  f0.0  x F!
       p_and_rho        F" x = x + dx"
       F" b = rho*dx^3"  ;

: done?   a F@ f1.0  F<=  ;   \ is Fermi energy <= mc^2 ?

: RESULTS                     \ display output
   b f@  MASS/M0  F*  fe.   x f@  R  F*  fe.  ;

: INTEGRATE    ( f: A[0] dA — )   \ solve equations
    INITIALIZE
    BEGIN   x_step  done?  UNTIL
    RESULTS ;

\ Say " f# central_fermi_energy  f# step-size  INTEGRATE "
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