Gravitation and Cosmology

Lecture 1:

Reading:

Ohanian, Ch. 1 — all!, Ch 2.1, 2.2

The principle of relativity

The principle of relativity was discovered by Galileo. It states that the laws of nature do not permit experimental measurement of absolute uniform motion. (The principle does not explain why this must be so.)

Consider two systems S and S′, moving uniformly (without acceleration) relative to each other. An experiment in S gives the same result as a similar experiment in S′. This is an experimental fact!

Moreover, if an observer in S measures something about some set of events (a firecracker exploding, *e.g.*), a comparably equipped observer in S′ will obtain exactly the same results about the same set of events.

Interestingly, absolute accelerated motion *can* be detected by a single observer. We experience this when we close our eyes and can feel ourselves pressed back in the seat cushions when an airplane takes off; or upon (re)swallowing our hearts when the plane falls freely in an air-pocket.

Lorentz transformation

The coordinate measurements in S and S′ are related by linear transformation:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}(\vec{v}) \tag{1.1}$$

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned} \tag{1.2}$$

The linear transformation coefficients can be derived assuming the speed of light is the same whether measured in S or S′. This is an *ex post facto* assumption (20-20 hindsight) based on the properties of Maxwell's equations.

To derive the transformation, assume the directions of the axes in S and S′ are aligned, and that the direction of relative motion is the x-direction (x^1). Then by symmetry,

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned} \tag{1.3}$$

or

$$\begin{aligned} \Lambda^{2}{}_{\nu} &= \delta^{2}{}_{\nu} \\ \Lambda^{3}{}_{\nu} &= \delta^{3}{}_{\nu} \end{aligned} \tag{1.3′}$$

How does symmetry predict this? Imagine holding up a standard meter stick in S, in the y or z direction, while your opposite number (ON) does the same in S′. You place some chalk at the ends of yours, and so does your ON. As the systems pass, the meter sticks brush each other. If yours is shorter, it will leave chalk on his; whereas if his is shorter, it will leave marks on yours.

But the distinction between $+x$ and $-x$ directions is a matter of convention (symmetry comes in here!). Thus if the principle of relativity is correct, this experiment cannot reveal which of you is moving, or in which direction. Hence the meter sticks must appear to be the same length.

Note that a crucial hidden point here is that both ends of the sticks could be compared simultaneously, as determined by clocks in either system.

Now what about comparing lengths along the direction of motion? As we just implied, one cannot be sure to compare simultaneously both ends of ON's meter stick with both ends of one in S, when they are lying along the x-direction.

Suppose both observers start their clocks when their origins coincide. Then S′ will measure your origin $x=0$ to be receding from him in his negative x direction at speed v. Its position will seem to be

$$x'_0 = -vt' \tag{1.4}$$

Now suppose one end of your meter stick is at a point x and the other at your origin in S. S′ will measure point x to be a distance X' from the origin of S, and receding from him at a speed v in the $-x$ direction. Hence S′ will say

$$x' = X' - vt' \,. \tag{1.5}$$

Now we cannot be sure $X' \equiv x$. That is, a meter stick in S′ aligned parallel to the direction of motion might appear elongated or shrunken compared with one in S. Thus write $x = \gamma X$, and so

$$x = \gamma\,(x' + vt') \tag{1.6a}$$

In principle, γ can depend on $|\vec{v}\,|$, but not on $\vec{v}$. (That is, on magnitude but not direction.)

By symmetry the only difference between S and S′ is the relative direction of motion, hence

$$x' = \gamma\,(x - vt) \tag{1.6b}$$

Thus we find

$$\Lambda^1{}_0 = -\gamma v/c \,. \tag{1.7}$$

Finally, we can write an expression for the time transformation:

$$x^{0\prime} \equiv ct' = \Lambda^0{}_0\, x^0 + \Lambda^0{}_1\, x^1 = \Lambda^0{}_0\, ct + \Lambda^0{}_1\, x \tag{1.8}$$

Comparing Eq. 1.6a,b we see that

$$\Lambda^0{}_0 = \gamma \tag{1.9a}$$

$$\Lambda^0{}_1 = \left(\frac{1}{\gamma} - \gamma\right)\frac{c}{v} \,. \tag{1.9b}$$

In Newton's view, it was obvious that $X' \equiv x$ or $\gamma = 1$, hence no time transformation was possible other than $t' \equiv t$.

But if we allow for $\gamma \neq 1$, we need to ask why there was a term in x, but no terms involving y or z in Eq. 1.8. We can see this by means of a simple thought experiment, based on the idea that c is the same in S and S′.

Suppose that when the origins of S and S′ coincided (at $t' = t = 0$) a flashbulb was set off at the origin. Then in each system, an observer will see an outwardly propagating spherical wave- front of light. The equation of the wave-front is

$$x^2 + y^2 + z^2 - (ct)^2 = 0 \tag{1.10}$$

in the S frame, and

$$x'^2 + y'^2 + z'^2 - (ct')^2 = 0 \tag{1.10′}$$

in S′. Since $y'^2 + z'^2 \equiv y^2 + z^2$, we may combine the equations to get

$$x^2 - (ct)^2 = x'^2 - (ct')^2 \,. \tag{1.11}$$

If t' depended on y or z as well as on t and x, then cross-terms in xy, xz, yt and zt would appear on the right hand side of Eq. 1.11, with no corresponding terms on the left. In other words, all the coefficients of such terms must be 0, which is the same thing as saying t' cannot depend on y or z.

Finally, by substituting x' from Eq. 1.6 and t' from Eq. 1.8 into Eq. 1.11, we find

$$\gamma^2 - c^2\left(\Lambda^0{}_1\right)^2 = 1 \qquad \text{(comparing } x^2 \text{ on both sides)}$$

$$\gamma^2\left(v^2 - c^2\right) = -c^2 \qquad \text{(comparing } t^2 \text{ on both sides)}$$

$$-2\gamma^2 v - 2c^2\gamma\Lambda^0{}_1 = 0 \qquad \text{(comparing } xt \text{ on both sides)}$$

From these equations we easily find

$$g = \frac{\pm 1}{\sqrt{1 - v^2/c^2}} \tag{1.12}$$

$$\Lambda^0{}_1 = -\gamma\, v/c \tag{1.13}$$

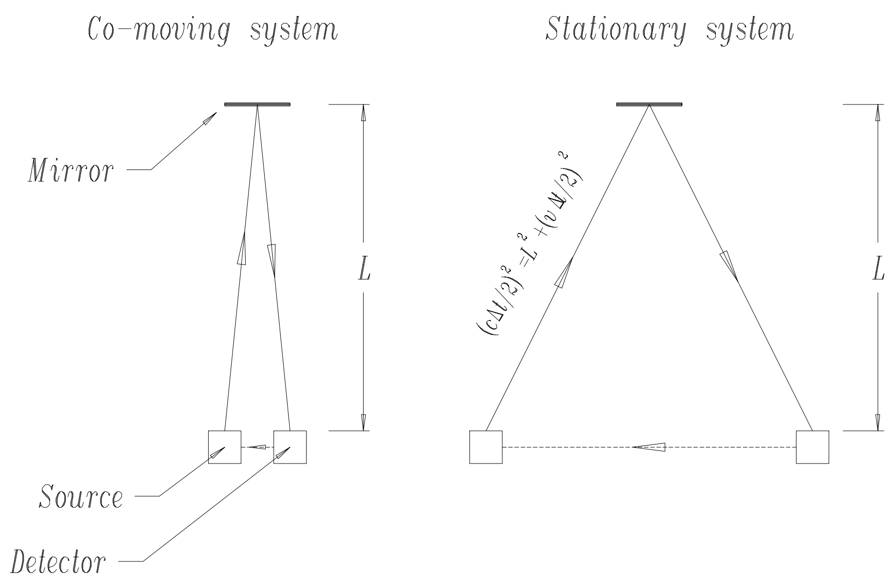
hence

$$t' = \gamma\left(t - \frac{vx}{c^2}\right). \tag{1.14}$$

Since we know experimentally that when v is small compared with c, t' is nearly the same as t (and increases monotonically with t), we must choose the positive root in Eq. 1.12.

Physical meaning of the Lorentz transformation

Consider a clock made out of light pulses that travel between a source, a mirror and a detector: the



source emits a short pulse of light that bounces off the mirror into the detector. When the detector receives the pulse it immediately triggers the source to emit another one.

The natural interval between ticks (in the rest frame of the clock, S′) is $\Delta t' = \frac{2L}{c}$. But an observer in S will determine the interval between ticks to be

$$\Delta t = \gamma \frac{2L}{c} \equiv \gamma \Delta t' \tag{1.15}$$

How would we derive this result from the Lorentz transformation? We see that the clock is stationary in S′, so $x'_{src} = x'_{det}$, and $\Delta t' = t'_{det} - t'_{src}$. From 1.14,

$$\Delta t = t_{det} - t_{src} = \gamma \left[t'_{det} - t'_{src} + \frac{v}{c}(x'_{det} - x'_{src}) \right] \tag{1.16}$$

$$\equiv \gamma \Delta t' ,$$

as we found by analyzing the above Figure.

Velocities near *c*

From the form of γ we see that something peculiar happens when $v = c$. In fact, the LT is not defined for $v \geq c$. Does this mean $v \geq c$ has no physical meaning, or is it a result of something wrong with our method? As we shall see below, it is the former: $v \geq c$ appears to have no physical interpretation, so we regard c as the limiting velocity for material objects and/or signal propagation.

Addition of velocities

The Lorentz transformation holds as well for small space and time intervals as for large ones. Thus suppose some object is moving in S with velocity u; in S′ it will appear to have velocity u', and we can use the Lorentz transformation to discover the relation between the two. The LT is

$$\begin{aligned} \delta x' &= \gamma\left(\delta x - v\delta t\right) \\ \delta t' &= \gamma\left(\delta t - v\delta t/c^2\right) \\ \delta y' &= \delta y \\ \delta z' &= \delta z \end{aligned} \qquad (1.17)$$

Thus, since observers in S and S′ must define

$$\begin{aligned} u_x &= \frac{\delta x}{\delta t} \\ u'_x &= \frac{\delta x'}{\delta t'} \end{aligned} \qquad (1.18)$$

respectively (and similarly for y- and z- components), we have

$$u'_x = \frac{\gamma\left(\delta x - \delta t\right)}{\gamma\left(\delta t - v\delta x/c^2\right)} = \frac{u_x - v}{1 - u_x v/c^2} \qquad (1.19a)$$

$$u'_y = \frac{\delta y}{\gamma\left(\delta t - v\delta x/c^2\right)} = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \qquad (1.19b)$$

$$u'_z = \frac{\delta z}{\gamma\left(\delta t - v\delta x/c^2\right)} = \frac{u_z}{\gamma\left(1 - u_x v/c^2\right)} \qquad (1.19c)$$

It is easy to see that if $v < c$ and $|\vec{u}\,| < c$, then $|\vec{u}\,'| < c$ also. That is, suppose I am in a rocket moving with speed $v = 3c/4$ with respect to the Earth, and launch another rocket moving with speed $u'_x = 3c/4$ with respect to me. The speed of the second rocket relative to the Earth will be (from Eq. 1.19a above)

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \equiv \frac{3c/4 + 3c/4}{1 + (3c/4)^2/c^2} = \frac{3c/2}{1 + 9/16} = \frac{24}{25}\,c\,. \qquad (1.20)$$

Thus we do not seem to be able to make something go faster than c by means of successive increments of velocity.

Aberration of starlight

Consider a star, and let its rest frame be S, and the Earth's rest frame be S′. Assume light from the star has velocity

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ -c \\ 0 \end{pmatrix}$$

What is the apparent velocity vector of the light in S′?

Assume the Earth moves in the *x*-*z* plane, in a circular orbit with

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v\cos\omega t' \\ 0 \\ v\sin\omega t' \end{pmatrix}$$

then the velocity components of the light beam are

$$\vec{u}'_{\perp} = -\vec{v}$$
$$\vec{u}'_{\parallel} = -\sqrt{c^2 - v^2}\ \hat{y}$$

<u>Note</u>: $|\vec{u}\,'| = c$, as can easily be checked.

Thus the star's image appears to trace out a circle as the earth moves in its orbit. The angular displacement of the star's image (the half- angle of the cone from the star to the circle) is

$$\tan\theta = \frac{u'_x}{u'_y} = \gamma v/c \approx v/c \qquad (1.21)$$

The aberration of starlight was once thought to be a decisive piece of evidence for a wave theory of light. Clearly it equally well fits a particle description, assuming the Lorentz transformation expresses general aspects of space-time not specific to electromagnetism alone.