Gravitation and Cosmology

Lecture 2:

Lorentz-invariant quantities

As we saw last time, the Lorentz transformation for our special case is

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma\,(t - vx/c^2) \\ \gamma\,(x - vt) \\ y \\ z \end{pmatrix} \qquad (2.1)$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

In general, the transformation from S to S' can be written as the product of a rotation and a *boost*. A boost is a transformation that applies to two systems with their axes aligned, moving with relative velocity $\vec{v}$. The general form of the transformation coefficients is

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v^1/c & -\gamma v^2/c & -\gamma v^2/c \\ -\gamma v^1/c & 1+(\gamma-1)\hat{v}^1\hat{v}^1 & (\gamma-1)\hat{v}^1\hat{v}^2 & (\gamma-1)\hat{v}^1\hat{v}^3 \\ -\gamma v^2/c & (\gamma-1)\hat{v}^2\hat{v}^1 & 1+(\gamma-1)\hat{v}^2\hat{v}^2 & (\gamma-1)\hat{v}^2\hat{v}^3 \\ -\gamma v^3/c & (\gamma-1)\hat{v}^3\hat{v}^1 & (\gamma-1)\hat{v}^3\hat{v}^2 & 1+(\gamma-1)\hat{v}^3\hat{v}^3 \end{pmatrix} \qquad (2.2)$$

Now, it is easy to see that the inverse transformation to $\Lambda^{\mu}{}_{\nu}(\vec{v})$ is $\Lambda^{\mu}{}_{\nu}(-\vec{v})$. That is,

$$\sum_{\kappa=0}^{3} \Lambda^{\mu}{}_{\kappa}(\vec{v})\ \Lambda^{\kappa}{}_{\nu}(-\vec{v}) = \delta^{\mu}{}_{\nu} \qquad (2.3)$$

(We will now drop the explicit Σ representing summations over repeated indices and use the Einstein summation convention that a repeated upper and lower index----like κ above----are summed from 0 to 3.)

Problem: Prove Eq. 2.3 by direct substitution of Eq. 2.2.

Now, by inspecting the special case Eq. 2.1 we see that the transformation closely resembles a rotation in a 4-dimensional space. One of the salient characteristics of a rotation is that it leaves lengths of vectors invariant. That is, ordinary 3-dimensional rotations do not affect the dot product

$$\vec{a}\cdot\vec{a} \equiv \left(a^1\right)^2 + \left(a^2\right)^2 + \left(a^3\right)^2 .$$

Similarly, the Lorentz transformation does not affect the "dot product"

$$-s^2 = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 \qquad (2.4)$$

That is,

$$-s'^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 = -s^2 ,$$

which the astute student will recognize as Eq. 1.11.

In a nutshell, if an observer in S measures the space-time coordinates ***x*** of an event and an observer in S′ measures the coordinates ***x*′** of the same event, and if they calculate $-s^2$ and $-s'^2$, respectively, their results will be numerically the same.

The easiest way to see the invariance of $-s^2$ is by direct substitution. For simplicity, confine attention to the special case Eq. 2.1; then since $y' = y$ and $z' = z$, we have only to be sure

$$(ct')^2 - (x')^2 = (ct)^2 - (x)^2 . \tag{1.11}$$

Of course this is correct because we used it to derive the Lorentz transformation in the first place!

Problem: demonstrate the invariance of $-s^2$ by direct substitution of the Lorentz transformation coefficients.

The coordinates ***x*** of a space-time event are actually a difference between two coordinates.

Problem: Why is the preceding remark correct?

Thus we can generalize the Lorentz-invariance of $-s^2$ to an infinitesimal interval between space-time points ***x*** and ***x*** + *d**x***:

$$(d\tau)^2 = \frac{1}{c^2}\left[\left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2\right] \equiv (dt)^2 - \vec{x}\cdot\vec{x}/c^2 \tag{2.5}$$

The infinitesimal Lorentz invariant quantity $d\tau$ is called the *proper time*. Its physical significance can be understood as follows: suppose a rocket moves at velocity $\vec{u}$ in the S system. We measure this velocity by measuring successive positions at successive ticks of a clock. Suppose the time-interval between ticks is dt. Then in time dt the rocket's position changes by $d\vec{x} = \vec{u}\,dt$. The proper time interval between successive position measurements is then

$$d\tau = \left((dt)^2 - \vec{x}\cdot\vec{x}/c^2\right)^{1/2} = dt\left(1 - \vec{u}\cdot\vec{u}/c^2\right)^{1/2} \tag{2.6}$$

Now consider a system S′ whose velocity $\vec{v}$ relative to S just happens to be the value of $\vec{u}$ at time t. Then as measured in S′ the rocket has velocity 0 and the (Lorentz invariant) proper time interval has the value dt'. In other words, the proper time is the time kept by the rocket pilot's own clock.

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Lecture 2:

Uniform acceleration in a fixed direction

Consider a rocket that----from the point of view of the passengers----has constant acceleration along the x-direction. That is, as measured in the rocket's own frame, in a time $d\tau$ (the time kept by the control- room clock) the rocket gains linear velocity

$$du = a d\tau \; . \tag{2.7}$$

What is the rocket's speed as seen from the frame S (*not* accelerating), with respect to which the rocket had speed 0 at τ=0?

At time τ the rocket had speed v, and at time $\tau + d\tau$ it has speed $v + dv$, in the S system. To find the new speed we use the formula for addition of velocities: in a frame S′ moving with velocity v in the x-direction, the rocket has (after time $d\tau$) speed $a d\tau$. (By taking $d\tau$ as small as we like, we can insure that the velocity du is extremely small compared with c.)

The speed in S is then

$$v + dv = \frac{v + du}{1 + v du/c^2} \approx (v + du)\,(1 - v du/c^2) \quad . \tag{2.8}$$

Expanding and keeping terms linear in du, we find

$$v + dv = v + du\,(1 - v^2/c^2) \; ,$$

or

$$dv = a d\tau\,(1 - v^2/c^2) \; . \tag{2.9}$$

This is a differential equation, that can be solved by separation of variables:

$$a\tau = \int_0^v dv'\,(1 - v'^2/c^2)^{-1} = \frac{c}{2}\log\left(\frac{1 + v/c}{1 - v/c}\right) \tag{2.10}$$

or

$$v(\tau) = c \tanh(a\tau/c) \; . \tag{2.11}$$

That is, as a function of ship time (*i.e.*, "proper" time), the velocity with respect to S increases from 0, but remains less than c. Its asymptotic value is c.

We would like now to relate the time t in S to the ship's time τ, so we can re-express the speed v as a function of t. Recall that

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} \; ,$$

so that

$$t = \int_0^\tau d\tau' \cosh(a\tau'/c) = \frac{c}{a}\sinh(a\tau/c) \; . \tag{2.12}$$

Therefore

$$v(t) = \frac{at}{\sqrt{1 + (at/c)^2}} \; . \tag{2.13}$$

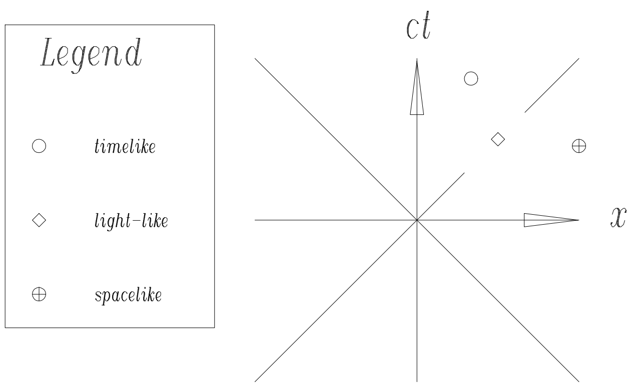
For small times, the speed is given by Newton's formula

$$v = at\ ;$$

but as time increases without limit, $v \rightarrow c$.

Physical meaning of s^2

The quantity s^2 defined previously is called the *invariant interval* between the origin in Σ and the spacetime event at $\boldsymbol{x}$. That is, if we think of the coinciding of the origins of S and S′ systems as a space-time event (event $\boldsymbol{0}$ in S), then the invariant interval represents something about the difference between the point $\boldsymbol{x}$ and the point $\boldsymbol{0}$.



The 45° lines represent the light cone, $x = ct$. The points represent events at timelike, lightlike or spacelike intervals from the origin

The physical interpretation is this:

- if $s^2 < 0$, then the interval is called *timelike*, and a light signal can connect the two events $\boldsymbol{O}$ and $\boldsymbol{x}$.
- if $s^2 = 0$, the interval is called *lightlike*.
- if $s^2 > 0$, the interval is called *spacelike* and the events $\boldsymbol{O}$ and $\boldsymbol{x}$ cannot be connected by a light signal.

What is this business about light signals? Basically it means that if something takes place at point $\vec{x}_A$ and time t_A , and something else takes place at $\vec{x}_B$ and a later time t_B , if someone could have sent a signal (by light beam, *e.g.*) from $\vec{x}_A$ at time t_A to point $\vec{x}_B$ and the signal could in principle have arrived *before* time t_B , then the event at $\vec{x}_A$ could have caused the event at $\vec{x}_B$. A simple calculation will show that in that case,

$$s_{AB}^2 = \left(\vec{x}_A - \vec{x}_B\right)^2 - c^2\left(t_A - t_B\right)^2 < 0\ .$$

Conversely, if the events are too far apart for a light signal to get from one to the other in time $\delta t = t_B - t_A$, then A could not possibly have caused B. In this case, $s_{AB}^2 > 0$. This is rather fortunate, because if $s_{AB}^2 > 0$, it would be possible for an observer---say in S---to think B occurred *after* A; while another observer---in S′, say---could determine that B occurred *before* A!