Gravitation and Cosmology
Lecture 3:

Scalars, vectors and tensors
We have examined Lorentz-invariant quantities s^2 and $d\tau$. In general, Lorentz invariant quantities are called *scalars* with respect to Lorentz transformations.

Similarly, the coordinates $x^\mu = (ct, \vec{x})$ form a 4-vector under Lorentz transformation. Any other set of four entities V^μ that transforms according to the law

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu \tag{3.1}$$

is also a 4-vector. Note that just because we stick indices on something it does not have to be a 4-vector. For example, the temperature, humidity, barometric pressure and time are four quantities, but because they do not transform according to Eq. 3.1, they are not a 4-vector. Conversely, the electromagnetic vector potential

$$A^\mu \equiv (\varphi, \vec{A})$$

(taking the Coulomb potential φ as the 0'th component) is a 4-vector.

We now consider objects with two indices (never mind what they are right now): if they transform according to the law

$$B'^{\mu\nu} = \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda B^{\kappa\lambda} \tag{3.2}$$

they are called second-rank tensors. And so, *ad infinitum*.

Note that a second-rank tensor can be written as the sum of a symmetric and an antisymmetric (or *skew-symmetric*) tensor:

$$B^{\mu\nu} \equiv \underbrace{\frac{1}{2}\left(B^{\mu\nu} + B^{\nu\mu}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(B^{\mu\nu} - B^{\nu\mu}\right)}_{\text{antisymmetric}}. \tag{3.3}$$

It is easy to see that in 4 dimensions, 6 numbers are needed to specify an antisymmetric 2nd-rank tensor, while 10 are needed to specify a symmetric 2nd-rank tensor. The total number of independent components needed to specify a general 2nd-rank tensor is of course $16 = 4\times4$.

So far we have considered tensors with their indices up. But another, related, transformation law is possible: Consider a scalar function of coordinates, $\varphi(x)$. clearly,

$$\varphi'(x') \equiv \varphi(x) \tag{3.4}$$

(the primes denote different frames, not differentiation).

Now, suppose we differentiate $\varphi(x)$ with respect to x^μ :

$$V_\mu = \frac{\partial\varphi(x)}{\partial x^\mu} \tag{3.5a}$$

Clearly, from the chain rule of calculus,

$$V'_\mu = \frac{\partial\varphi'(x')}{\partial x'^\mu} = \frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial\varphi(x)}{\partial x^\kappa} \tag{3.5b}$$

$$V'_\mu = \frac{\partial x^\kappa}{\partial x'^\mu} V_\kappa \equiv \left[\Lambda^{-1}\right]^\kappa{}_\mu \; V_\kappa \qquad (3.5b')$$

where we have used the fact that if

$$dx'^\mu = \Lambda^\mu{}_\nu \, dx^\nu \, ,$$

then

$$dx^\kappa = \left[\Lambda^{-1}\right]^\kappa{}_\mu \; dx'^\mu \quad . \qquad (3.6)$$

Tensors with their indices up are called *contravariant* tensors, whereas those with indices down are called *covariant*.

The Minkowski tensor

We can express $(d\tau)^2$ as

$$(d\tau)^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu \qquad (3.7)$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (3.8)$$

is called the *Minkowski tensor* (a special case of a *metric tensor*).

Clearly, $\eta_{\mu\nu}$ transforms as a covariant tensor of rank two:

$$(d\tau')^2 = \eta'_{\mu\nu} \, dx'^\mu \, dx'^\nu = \eta'_{\mu\nu} \, \Lambda^\mu{}_\kappa \, dx^\kappa \; \Lambda^\nu{}_\lambda \, dx^\lambda = \eta_{\kappa\lambda} \, dx^\kappa \, dx^\lambda = (d\tau)^2 \qquad (3.9)$$

That is, comparing coefficients of $dx^\kappa \, dx^\lambda$, we find

$$\eta_{\kappa\lambda} \, dx^\kappa \, dx^\lambda = \eta'_{\mu\nu} \; \Lambda^\mu{}_\kappa \; \Lambda^\nu{}_\lambda \qquad (3.10)$$

but this is just the statement that

$$\eta'_{\mu\nu} = \left[\Lambda^{-1}\right]^\kappa{}_\mu \left[\Lambda^{-1}\right]^\lambda{}_\nu \; \eta_{\kappa\lambda} \qquad (3.11)$$

i.e. $\eta_{\mu\nu}$ is a covariant tensor under Lorentz transformation.

We can use $\eta_{\mu\nu}$ to convert a contravariant tensor to a covariant one:

$$V_\mu = \eta_{\mu\nu} \, V^\nu \qquad (3.12)$$

Clearly, V_μ in Eq. 3.12 is a covariant vector if V^ν is a contravariant one.

Gravitation and Cosmology

Lecture 3:

The Thomas precession

In 1926, L.H. Thomas showed that the spin axis of a spinning object precesses (relative to an unaccelerated frame) if the object is accelerated by some external force---even if there is no net torque acting on the spin! This effect is purely relativistic, and is called the Thomas precession in his honor. We now derive it.

In the instantaneous rest frame of the object (IRF), which at time t has velocity $\vec{v}$ with respect to the lab frame S, in the absence of torques the spin is always a vector with components

$$s^{\mu} = (0, \vec{s}\,) \tag{3.13}$$

In the lab frame S, the particle has spin 4-vector

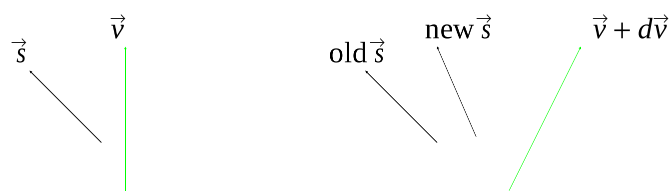
$$S^{\mu} = \Lambda^{\mu}{}_{\nu}\,(-\vec{v}\,)\, s^{\nu} \,.$$

At time $t + dt$ the particle has velocity $\vec{v} + d\vec{v}$ with respect to S, and its IRF is S′ which is reached from the lab frame by the boost $\Lambda^{\mu}{}_{\nu}\,(\vec{v} + d\vec{v}\,)$.

To compare a vector with another, we must do so in the same frame. Thus we take the vector

$$S'^{\mu} = \Lambda^{\mu}{}_{\kappa}\,(\vec{v} + d\vec{v}\,)\, \Lambda^{\kappa}{}_{\nu}\,(-\vec{v}\,)\, s^{\nu}$$

and compare it with s^{μ}, the spin in the IRF. This is illustrated in the Figure below:



The effective change of the spin in the IRF is thus given by

$$ds^{\mu} = \Lambda^{\mu}{}_{\kappa}\,(\vec{v} + d\vec{v}\,)\, \Lambda^{\kappa}{}_{\nu}\,(-\vec{v}\,)\, s^{\nu} - s^{\mu} \tag{3.14}$$

using Eq. 2.2 we find, in the limit of small v/c :

$$d\vec{s} = \frac{1}{2mc^2}\,\vec{s} \times (\vec{v} \times d\vec{v}\,) \tag{3.15}$$

which, from Newton's 3rd law (valid for $v/c << 1$) gives the rotation rate of $\vec{s}$:

$$\frac{d\vec{s}}{dt} \approx \frac{1}{2mc^2}\,\vec{s} \times (\vec{v} \times \vec{F}) \tag{3.16}$$

From Hamiltonian dynamics, we see that if the force is derived from a potential $\vec{F} = -\nabla V$, the effective torque is equivalent to an extra term in the Hamiltonian:

$$\Delta H = \frac{1}{2mc^2}\,\vec{s} \cdot (\vec{s} \times \nabla V) \,. \tag{3.17}$$

If the force is central, $\nabla V(r) \equiv \frac{\vec{r}}{r}\frac{dV(r)}{dr}$, and if we approximate $\vec{v}$ by $\frac{\vec{p}}{m}$, we see that in terms of the orbital angular momentum, $\overrightarrow{L} = \vec{r} \times \vec{p}$,

$$\Delta H_{Thomas} = \frac{1}{2mc^2 r}\frac{dV(r)}{dr}\vec{L}\cdot\vec{s}\ . \tag{3.18}$$