
Dynamics of Particles

Relativistic momentum

The Newtonian definition of momentum is

$$\vec{p} = m \frac{d\vec{x}}{dt} \quad (4.1)$$

Eq. 4.1 transforms like a 3-vector under Galilean transformations.

But Eq. 4.1 does not transform like a vector under Lorentz transformations. That is, the above definition does *not* obey

$$p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}. \quad (4.2)$$

This will lead to momentum nonconservation in some frames.

One way out is to note that what went wrong is dt is not a Lorentz scalar (although it *is* a Galilean scalar). One can easily construct a 4-vector that reduces to the Newtonian definition at small velocities by using $d\tau$ rather than dt :

$$p^{\mu} = m \frac{dx^{\mu}}{d\tau}. \quad (4.3)$$

We see that in terms of $\vec{u} = \frac{d\vec{x}}{dt}$, the space components of p^{μ} are

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}. \quad (4.4)$$

We shall discuss the time component, $p^0 = \frac{mc}{\sqrt{1 - u^2/c^2}}$, shortly.

The above has been a mathematician's approach to guessing the correct form of the momentum in Special Relativity. Is there some way a physicist might deduce it? Obviously there is, or I wouldn't bring up the question!

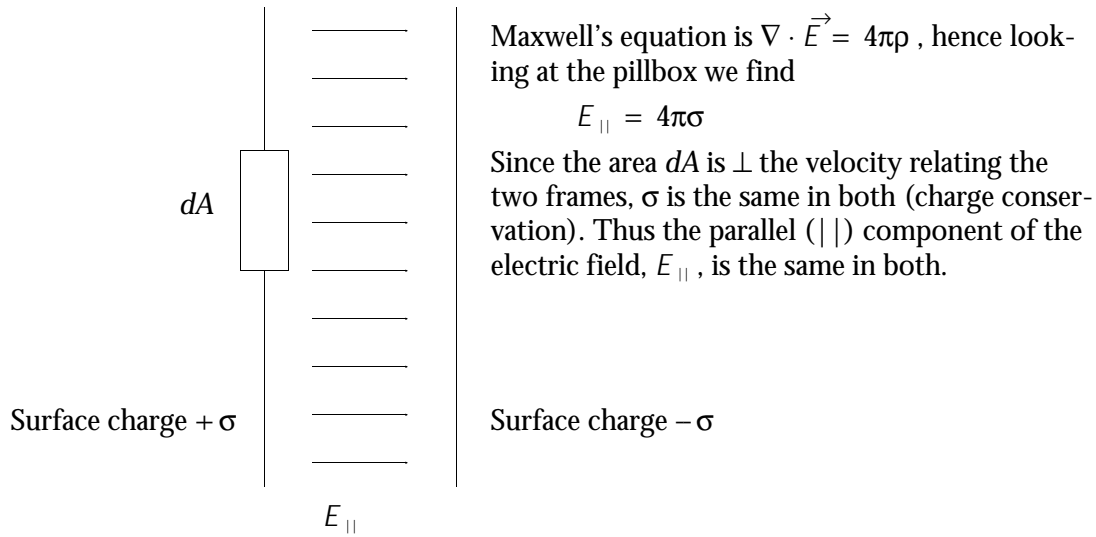
We want the momentum to obey Newton's Law,

$$\frac{d\vec{p}}{dt} = \vec{F}.$$

We also want to have $\vec{p} \approx m\vec{u}$ at small speeds, $u \ll c$. We can also use what we know about the relativistic law of transformation of velocities. And finally, we can use common sense to derive the transformation law of an electric field parallel to the direction of the Lorentz boost. Let us do this first, to get it out of the way. The essentials are shown below in Fig. 4.1.

Gravitation and Cosmology

Relativistic momentum



Consider a test particle of charge q : In the lab (S) its equation of motion is (in the direction)

$$dp = qE_{||} dt \quad (4.5)$$

whereas in the moving system (instantaneous rest frame) we have

$$dp' = qE'_{||} dt' = qE_{||} dt' = qE_{||} d\tau \quad (4.5')$$

Hence

$$qE_{||} = \frac{dp}{dt} = \frac{dp'}{d\tau}$$

or

$$dp = dp' \frac{dt}{d\tau} = \frac{dp'}{\sqrt{1 - u^2/c^2}} \quad (4.6)$$

In the instantaneous rest frame we have

$$dp' = m du'$$

while from addition of velocities we have

$$u + du = \frac{u + du'}{1 + u du'/c^2}$$

or

$$du = du' (1 - u^2/c^2) \quad (4.7)$$

Thus from Eq. 4.6 we have

$$dp = m du (1 - u^2/c^2)^{-3/2} \quad (4.8)$$

That is, our assumptions have led to a differential equation for linear momentum in terms of linear velocity. Calling $\beta = u/c$ we have

Gravitation and Cosmology

Lecture 4: Dynamics of Particles

$$p = mc \int_0^\beta d\beta (1 - \beta^2)^{-3/2} = \frac{mc\beta}{\sqrt{1 - \beta^2}} \quad (4.9)$$

This strongly suggests that

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}} \quad (4.10)$$

is indeed the correct generalization of Newtonian momentum.

Problem:

Show explicitly that if momentum is conserved in an inertial frame S, then according to Eq. 4.10 it is conserved in a frame S' related to S by a Lorentz transformation.

(Hint: use the relativistic transformation of velocity.)

Energy

Again, we would like to keep the physical laws in Special Relativity as close as possible to the Newtonian form. This will guarantee no surprises re energy conservation, *etc.*

Thus we define energy as work:

$$dW = \vec{F} \cdot d\vec{x} \quad (4.11)$$

or

$$\begin{aligned} W &= \int d\vec{x} \cdot \frac{d\vec{p}}{dt} = \int \frac{d\vec{x}}{dt} \cdot \frac{d\vec{p}}{dt} dt = \int \vec{u} \cdot \frac{d\vec{p}}{dt} dt = \int \vec{u} \cdot d\vec{p} \\ &= \int m\vec{u} \cdot d\vec{u} (1 - u^2/c^2)^{-3/2} \end{aligned} \quad (4.12)$$

$$\therefore W = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2 \quad (4.13)$$

The energy gained in accelerating a particle from $u = 0$ to some final speed u reduces to

$$\Delta W \approx \frac{1}{2} mu^2 \quad (4.14)$$

which is the Newtonian kinetic energy. Since the energy gain is the difference of two energies,

$\frac{mc^2}{\sqrt{1 - u^2/c^2}}$ and mc^2 , this result strongly suggests we interpret the energy of a moving particle to be

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}.$$

It further says that the energy of a particle at rest is mc^2 . This has implications we shall consider in the next lecture.

Gravitation and Cosmology

Energy

But for now, note that the p^0 we found as $p^0 = \frac{mc}{\sqrt{1 - u^2/c^2}}$ is related to E via

$$E = p^0 c \tag{4.15}$$

That is, we can interpret the 4th component of 4-momentum as energy, and we see that the conservation of energy---far from being a law distinct from conservation of momentum---actually is part of the same law: Conservation of 4-momentum.