Weighty matters

Lorentz invariants (cont'd)

We saw that a covariant vector transforms as

$$V'_{\mu} = \left[\Lambda^{-1}\right]^{\nu}_{\ \mu} V_{\nu} \tag{5.1}$$

and a contravariant one as

$$p'^{\mu} = \Lambda^{\mu}_{\ \nu}\, p^{\nu}\,. \tag{5.2}$$

This means that the *contraction* $V_{\mu}\, p^{\mu}$ of a covariant with a contravariant vector is a scalar under Lorentz transformation. In general this result can be extended to the contraction of any set of covariant with a corresponding set of contravariant indices.

Relativistic energy (cont'd)

We recall the formula for the energy of a moving particle:

$$E = \frac{mc^2}{\sqrt{1 - \vec{u}^2/c^2}} \tag{5.3}$$

Suppose we evaluate this for small velocities: we get

$$E \approx mc^2 \left(1 + \frac{1}{2}\frac{\vec{u}^2}{c^2} + \ldots\right) = mc^2 + \frac{1}{2} m\vec{u}^2 + \ldots \tag{5.4}$$

Except for the extra term mc , Eq. 5.4 is the Newtonian kinetic energy.

The meaning of $E = mc^2$

It is easy to see that the *rest-energy*, mc^2 , is Lorentz invariant. Manifestly, the contraction

$$p^{\mu} p_{\mu} = \left(p^0\right)^2 - \vec{p}\cdot\vec{p}$$

is Lorentz invariant; but from our previous results,

$$\left(p^0\right)^2 - \vec{p}\cdot\vec{p} = \frac{(mc)^2 - m^2\, \vec{u}\cdot\vec{u}}{1 - \vec{u}^2/c^2} = (mc)^2\,. \tag{5.5}$$

That is, the rest-energy can be expressed as a contraction of a 4-vector with itself, hence is necessarily Lorentz invariant.

Now, since the mass m appears as the coefficient of $\frac{1}{2}\vec{u}^2$ in the Newtonian kinetic energy, it must be (this is an *experimental* fact, accurate to about one part in 10^{13}) the very same mass that appears in the Newtonian gravitational force $\vec{F}_{grav} = \dfrac{-GmM\,\vec{r}}{r^3}$.

Put somewhat differently, the total energy $E = mc^2$ of a body at rest can be weighed because the m is the same as that appearing in the formula for weight at the Earth's surface:

$$F_{grav} = -mg \ . \qquad (5.6)$$

Now what, precisely, does it mean to say we can weigh energy? The only consistent interpretation is that it means exactly what it says: all energy, any kind of energy, the total energy content of an object, contribute to its weight.

For example, consider a storage battery. Its energy content is greater when it is charged than when it is discharged. Hence it must be heavier, when charged, by about 10^{-9} Nt. This is a fractional change in mass that would be very difficult to measure, because of its insignificance.

Problem:
Given that a 12 V car battery masses about 20 Kg and holds a charge of 150 Ampere-hours (that is, it can discharge at 1 amp for 150 hours or 150 amps for 1 hour), show that the fractional change in weight is roughly that given above.

However, the forces in nuclear and particle physics are very strong on the scale of the masses of the nuclear constituents (hadrons) so we can estimate that the mass change associated with a typical nuclear energy----for example, the ≈ 5 MeV released in a typical a-particle decay:

$$Parent \rightarrow Daughter + \alpha + 5\ \text{MeV}$$

of a heavy nucleus----is large enough for us to measure the corresponding mass change. We see that, by generalized energy conservation,

$$M_{parent}\, c^2 = M_{daughter}\, c^2 + m_{\alpha}\, c^2 + 5\ \text{MeV} \ . \qquad (5.7)$$

The relative mass change, in ^{226}Ra, is about

$$\frac{\delta M}{M} \approx \frac{5\ \text{MeV}}{226 \times 940\ \text{MeV}} = 2.4\times10^{-5} \ . \qquad (5.8)$$

Interestingly, in 1905 Einstein suggested measuring the mass changes of radioactive nuclei as an experimental test of $E = mc^2$, but the measurement was not actually carried out until 1933.

All kinds of energy have gravitational mass

The following argument is due to Einstein in its essentials. He used it to prove that photons have a gravitational interaction. The proof works by contradiction:

Suppose there is some form of energy (perhaps a new, unknown kind called unclear energy, or UE) that does not weigh anything.

Then we can use this property of UE to produce a perpetual motion machine. This contradicts conservation of energy, so the premise that UE has gravitational mass different from

$$m = E/c^2$$

---either heavier or lighter---must be wrong.

Here is the proof:

We need a rope and pulley connected to a generator that can charge up a capacitor. We suppose we have the unclear equivalent, an uncleator that can be charged with UE (using a perfect electrical to UE energy converter ---the perverter). Assume that when discharged the uncleator (U) and capacitor (C) weigh the same. We suppose we have 2 C's and 2 U's. The energy in the charge is δmc^2.

1. Start with a discharged C and a charged one (C*) at the top, and 2 discharged U's at the bottom. Put the C on at the top, and a U on at the bottom. The C now falls a distance h, raising the U by h, and doing work $W_1 = gh\delta m$.
2. We now have CU at the top, and C* U at the bottom. Now discharge C* into U at the bottom, making a U* that weighs the same as U. Raise U* and lower U. This takes work $W_2 = 0$.
3. We now have CU* at the top and CU at the bottom. Discharge U* into C at the top, leaving C*U at the top, CU at the bottom. This takes work $W_3 = 0$.
4. Raise C and lower U. This takes work $W_4 = 0$. The system now has C*C at the top, UU at the bottom, as at the beginning of Step 1.

We have a system that---upon completing one cycle---has produced net work

$$W_1 + W_2 + W_3 + W_4 = gh\delta m ,$$

and has returned to its initial state. This is the definition of perpetual motion!

Gravitation and Cosmology

All kinds of energy have gravitational mass