The energy-momentum tensor

Reading: Ohanian, §2.4, 2.5

In this Chapter and subsequently, we shall follow the convention $c=1$.

Currents

The equation of current conservation (electrical, particle number, probability or whatever) is

$$\partial\rho + \nabla\cdot\vec{j} = 0 \,. \tag{6.1}$$

Written in 4-dimensional notation, this is

$$\partial_\mu J^\mu = 0 \,, \tag{6.2}$$

where $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$. Because it is the contraction of 2 tensor indices, and because we already know that ∂_μ transforms as a covariant 4-vector), we see that Eq. 6.1 would be manifestly Lorentz invariant if J^μ were a contravariant 4-vector (more precisely, a vector density; but we shall leave this detail for a future lecture).

Of course, something as basic as a conservation law must be Lorentz invariant---that is, it cannot depend on the frame within which we make our observations. To prove it is so we must determine that J^μ is indeed a 4-vector.

Consider the density for a point particle located at $\vec{\xi}\,(t)$:

$$\rho(\vec{r}\,, t) = \delta(\vec{r} - \vec{\xi}\,(t)) \tag{6.3}$$

Its time derivative is

$$\partial_t\,\rho = -\frac{d\vec{\xi}}{dt}\cdot\nabla\delta(\vec{r} - \vec{\xi}\,(t)) = -\nabla\cdot\left[\frac{d\vec{\xi}}{dt}\,\delta(\vec{r} - \vec{\xi}\,(t))\right] \tag{6.4}$$

or

$$\vec{j}\,(\vec{r}\,, t) = \delta(\vec{r} - \vec{\xi}\,(t))\,\frac{d\vec{\xi}}{dt} \equiv \delta(\vec{r} - \vec{\xi}\,(t))\,\vec{u}(t) \,. \tag{6.5}$$

Now, as we have seen, it is possible to define the proper time τ as a function of *t*---and *vice versa*---by integrating the equation

$$d\tau = dt\sqrt{1-\vec{u}^2} \quad , \tag{6.6}$$

so, defining $\xi^0(\tau) = t(\tau)$, we can rewrite ρ and $\vec{j}$ in the combined form

$$J^\mu(\vec{r}\,,\ t) = \int d\tau\ \delta^{(3)}(\vec{r} - \vec{\xi}(t))\,\delta(t-\xi^0(\tau))\,\frac{d\xi^\mu(\tau)}{d\tau} \,. \tag{6.7}$$

Since $d\tau$ is manifestly a scalar under Lorentz transformation, and since $\frac{d\xi^\mu(\tau)}{d\tau}$ is manifestly a contravariant 4-vector, it remains only to show that

$$\delta^{(4)}\Big(x^\mu - \xi^\mu(\tau)\Big) \equiv \delta^{(3)}(\vec{r} - \vec{\xi}(t))\ \delta(t - \xi^0(\tau))$$

is a Lorentz invariant density.

To do this we note that

$$\int d^4x\ \delta^{(4)}\Big(x^\mu - \xi^\mu(\tau)\Big) = 1 ,$$

(which is the same in any coordinate system!) so all we have left to show is that the 4-dimensional volume element d^4x is Lorentz invariant. But this is easy: in our special case,

$$d^4x = dy\,dz\,dx\,dt$$
$$d^4x' = dy'\ dz'\ dx'\ dt' = dy\,dz\,dx'\ dt' .$$

Hence we must show $dx\,dt = dx'\ dt'$.

Now, when changing variables of integration, we may write

$$dx'\ dt' = \left|\frac{\partial(x',t')}{\partial(x,t)}\right| dx\,dt . \tag{6.8}$$

The Jacobian, of the transformation

$$\left|\frac{\partial(x',t')}{\partial(x,t)}\right| = \det\begin{bmatrix}\partial x'/\partial x & \partial x'/\partial t\\ \partial t'/\partial x & \partial t'/\partial t\end{bmatrix} = \det\begin{bmatrix}\gamma & -\gamma v\\ -\gamma v & \gamma\end{bmatrix} \tag{6.9}$$

$$= \gamma^2\,(1 - v^2) = 1 .$$

Hence, $dx'\ dt' = dx\,dt$, and so J^μ is indeed a 4-vector with respect to Lorentz transformations.

Energy-momentum tensor

We now consider the object (for a point particle of mass m)

$$T^{\mu\nu}(\vec{r},\ t) = m\int d\tau\ \delta^{(4)}\Big(x^\mu - \xi^\mu(\tau)\Big)\,\frac{d\xi^\mu(\tau)}{d\tau}\,\frac{d\xi^\nu(\tau)}{d\tau} . \tag{6.10}$$

Manifestly, again, this is a second-rank contravariant tensor (density) with respect to Lorentz transformation. We now want to consider its physical meaning.

First of all, we see that

$$\partial_\mu T^{\mu\nu}(\vec{r}\,,\; t) = m \int d\tau \; \frac{d\xi^\mu(\tau)}{d\tau} \frac{d\xi^\nu(\tau)}{d\tau} \; \partial_\mu \, \delta^{(4)}\Big(x^\mu - \xi^\mu(\tau)\Big)$$

$$= -m \int d\tau \; \frac{d\xi^\nu(\tau)}{d\tau} \frac{d}{d\tau} \delta^{(4)}\Big(x^\mu - \xi^\mu(\tau)\Big) \qquad (6.11)$$

or, since for a free particle, $\frac{d^2\xi^\mu(\tau)}{d\tau^2} = 0$, we find upon integrating by parts and discarding the end-point contribution,

$$\partial_\mu T^{\mu\nu} = m \int d\tau \; \delta^{(4)}\Big(x^\mu - \xi^\mu(\tau)\Big) \frac{d^2\xi^\nu(\tau)}{d\tau^2} = 0\,. \qquad (6.12)$$

That is, $T^{\mu\nu}$ is conserved.

Clearly, $T^{\mu 0}$ is a conserved 4-vector density. If we integrate it over volume (over all space) we obtain

$$\int d^3x \;\; T^{\mu 0} = m \frac{d\xi^\mu}{d\tau} = p^\mu\,. \qquad (6.13)$$

Hence we can interpret $T^{\mu 0}$ as the 4-momentum density of a point particle.

The tensor $T^{\mu\nu}$ is called the energy-momentum tensor. It is symmetric in $\mu\nu$.

4-momentum density of a gas

The energy-momentum tensor of a collection of non-interacting point particles is

$$T^{\mu\nu}(\vec{r}\,,\; t) = \sum_{k=0}^{N} \delta^{(3)}(\vec{r} - \vec{\xi}_k(t)) \; \frac{p_k^\mu \, p_k^\nu}{E_k} \qquad (6.14)$$

On the other hand, the energy momentum tensor of a perfect fluid has the form[†]

$$T^{\mu\nu} = U^\mu \, U^\nu \, (\rho + p) - p \, \eta^{\mu\nu} \qquad (6.15)$$

where U^μ is the 4-velocity of the rest-frame of the fluid with respect to the observer's frame (the "Laboratory"). The parameters ρ and p are the "proper" energy density and pressure, respectively.

This means that for a relativistic perfect gas in its rest frame,

$$\begin{aligned} T^{ij} &= p\,\delta_{ij} \\ T^{00} &= \rho \end{aligned} \qquad (6.16)$$

and therefore, in this frame,

† See Ohanian and Ruffini, 2nd. ed., prob. 2.28.

$$p = \frac{1}{3} \sum_{k=0}^{N} \delta^{(3)}(\vec{r} - \vec{\xi}_k(t)) \frac{\vec{p}_k \cdot \vec{p}_k}{E_k} \quad , \tag{6.17}$$

$$\rho = \sum_{k=0}^{N} \delta^{(3)}(\vec{r} - \vec{\xi}_k(t)) \; E_k \, . \tag{6.18}$$

For any gas, $p \le \frac{1}{3}\rho$; for an ultra-relativistic gas (for example, a gas of photons)

$$p = \tfrac{1}{3}\rho \, . \tag{6.19}$$

Sound in an ultrarelativistic gas

Eq. 6.19 has the following interesting consequence† for the propagation of sound: from the second law of thermodynamics, if n is the proper particle-number density and σ is the proper entropy, then

$$kTd\sigma = p\, d\left(\frac{1}{n}\right) + d\left(\frac{\rho}{n}\right), \tag{6.20}$$

where k is Boltzmann's constant. We consider small disturbances $\delta\rho$, δp, δn and $\delta\vec{v}$ to the average values of ρ, n, p, and $\vec{v}$ (=0). The conservation of particle number gives

$$\partial_t \, \delta n + n \nabla \cdot \delta\vec{v} = 0 \tag{6.21}$$

A sound wave involves adiabatic compression, hence no change in entropy. Thus

$$-p\delta n + n\delta\rho - \rho\delta n = 0 \, . \tag{6.22}$$

But we also have, from $\partial_\mu T^{\mu\nu} = 0$ (keeping only terms to first order in $\delta\vec{v}$) that

$$\partial_t \, \delta\vec{v} + \frac{\nabla p}{p + \rho} = 0 \, . \tag{6.23}$$

Now, supposing that $\delta\rho$ (the change in internal energy) $\approx \lambda\, \delta p$, we have

$$\partial_t \, \delta\vec{v} + \frac{1}{\lambda}\frac{\nabla\rho}{p+\rho} = \partial_t \, \delta\vec{v} + \frac{1}{\lambda}\nabla\left(\frac{\delta n}{n}\right) = 0 \tag{6.24}$$

which, together with Eq. 6.21, yields the wave equation

$$\frac{\partial^2}{\partial t^2}\delta n - \frac{1}{\lambda}\nabla^2 \, \delta n = 0 \, . \tag{6.25}$$

We can therefore interpret the square of the sound velocity as λ^{-1} (in units of c). For a gas of ultrarelativistic particles, $\lambda = 3$, hence

$$u_{sound}/c = \sqrt{1/3} \, . \tag{6.26}$$

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† See, *e.g.*, Weinberg, §2.10.