The variational approach to dynamics

Lightning review of nonrelativistic mechanics

Hamilton's principle of Least Action (PLA):

$$\mathsf{A}\,(\{q\}) = \int_{t_A}^{t_B} dt\, L(q, \dot{q}, t) \tag{7.1}$$

A is a *functional* of $q(t)$, where $q(t)$ stands for all the (generalized) coordinates of a physical system.

The Principle of Least Action states that

$$\frac{\delta \mathsf{A}}{\delta q(t)} = 0 \tag{7.2}$$

i.e. the physical trajectory makes the action stationary with respect to small deviations $\delta q(t)$ about the physical trajectory, subject to

$$\delta q(t_A) = \delta q(t_B) = 0\,. \tag{7.3}$$

The function $L(q, \dot{q}, t)$ is called the *Lagrangian* of the system. If we know the Lagrangian we can in principle know all there is to know about a system.

Digression

How does the PLA work? We calculate the difference

$$\delta \mathsf{A} = \mathsf{A}\,(\{q + \delta q\}) - \mathsf{A}\,(\{q\}) = \int_{t_A}^{t_B} dt \left[L(q + \delta q, \dot{q} + \tfrac{d}{dt}\delta q, t) - L(q, \dot{q}, t) \right]$$

$$= \int_{t_A}^{t_B} dt \left[\delta q(t) \frac{\partial L}{\partial q} + \tfrac{d}{dt}\delta q(t) \frac{\partial L}{\partial \dot{q}} \right]. \tag{7.4}$$

Upon integrating the term in $\tfrac{d}{dt}\delta q(t)$ once by parts we find

$$\delta \mathsf{A} = \int_{t_A}^{t_B} dt\, \delta q(t) \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] + \delta q(t) \left(\frac{\partial L}{\partial \dot{q}}\right)\Bigg|_{t_A}^{t_B}$$

$$= \int_{t_A}^{t_B} dt\, \delta q(t) \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right], \tag{7.5}$$

where we have used Eq. 7.3 to drop the end point contribution.

Since $\delta q(t)$ is, subject to the constraint Eq. 7.3, an arbitrary function, in order for the integral in Eq. 7.5 to vanish, the integrand must vanish, *i.e.*

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 . \tag{7.6}$$

Equation 7.6 is known as the Euler-Lagrange equation. If there is more than one generalized coordinate---for example, a particle moving in 3-dimensional space will be described by three position coordinates $\left(x(t), y(t), z(t)\right)$---there is an Euler-Lagrange equation arising from the (independent) variation of each coordinate, making three in this case.

End of digression

The Lagrangian of a nonrelativistic particle is

$$L = \frac{1}{2} m\vec{v}^2 - V(\vec{x}) \tag{7.7}$$

for which the (three) Euler-Lagrange equations yield Newton's 3rd Law:

$$m \frac{d\vec{v}}{dt} = -\nabla V(\vec{x}) . \tag{7.8}$$

Lorentz invariant mechanics

The Lagrangian of a free particle must be such as to give rise to a scalar action. In the nonrelativistic case, the action is indeed a scalar under Galilean transformations, hence the resulting equations of motion respect the Galilean Principle of Relativity. Clearly, if we wish our equations of motion to respect the Principle of Relativity as formulated by Einstein (that is, with respect to Lorentz transformations), we had better choose an action that is a Lorentz scalar. This requirement is actually very restricting. For a free particle, the only scalar quantity is the proper time,

$$\tau = \int dt \sqrt{1 - \vec{v}^2/c^2} .$$

Thus we choose for our Lagrangian

$$L = -mc^2 \sqrt{1 - \vec{v}^2/c^2} \tag{7.9}$$

since this produces the (manifestly) Lorentz invariant action

$$\mathsf{A} = \int dt\, L = -mc^2 \int dt \sqrt{1 - \vec{v}^2/c^2} \equiv -mc^2 \int d\tau . \tag{7.10}$$

Now, what about a particle that is not free: how can we express potential energy in a Lorentz invariant manner?

Clearly, we must add to $-mc^2$ quantities that are scalars with respect to Lorentz transformation, but functions of the coordinates and/or velocity of the particle. Such quantities would have the form

$$\varphi(x)$$
$$u^\mu A_\mu (x)$$
$$u^\mu u^\nu B_{\mu\nu} (x)$$
$$\ldots$$

where φ is a Lorentz scalar, A_μ a covariant 4-vector, *etc.*

Looking ahead for a moment, consider the electromagnetic 4-vector potential, $A^\mu = (\varphi, \vec{A})$. For the moment, let us work in a frame where $\vec{A} = 0$. We know that, nonrelativistically, the Lagrangian takes the form

$$L = -mc^2 + \frac{1}{2} m\vec{v}^2 - Q\,\varphi(\vec{x}(t)) \tag{7.11}$$

(*i.e.*, if we did not include the $-mc^2$, it would just be the familiar $T - V$).

But

$$Q\,\varphi\,dt = Q\,\varphi \frac{dt}{d\tau} d\tau \equiv Q\,A_0\,u^0\,d\tau \tag{7.12}$$

so the proper Lorentz invariant form of the potential energy must be

$$Vdt = Q\left(\varphi - \vec{v}\cdot\vec{A}\right)dt = Q\,u^\mu A_\mu\,d\tau\,. \tag{7.13}$$

The Euler-Lagrange equations are then obtained by varying with respect to x^μ subject to the (holonomic) constraint that $u^\mu u_\mu = 1$.

The relativistic Lagrangian of a charged point particle in a 4-vector potential is then

$$L = -m\sqrt{u^\mu u_\mu} - Q\,u^\mu A_\mu$$

so

$$\frac{\partial L}{\partial u^\mu} = -Q\,A_\mu - \frac{m u_\mu}{\sqrt{u^\mu u_\mu}}$$

and

$$\frac{\partial L}{\partial x^\mu} = -Q\,u^\nu\,\partial_\mu A_\nu\,,$$

giving the Euler-Lagrange equation

$$-Q\,u^\nu\,\partial_\mu A_\nu - \frac{d}{d\tau}\left(-Q\,A_\mu - \frac{m u_\mu}{\sqrt{u^\mu u_\mu}}\right) = 0\,,$$

or since $\frac{d}{d\tau} u^\mu u_\mu = 0$ (because of the constraint $u^\mu u_\mu = 1$)

$$m\frac{du_\mu}{d\tau} = Q\,u^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right). \tag{7.14}$$

Eq. 7.14 is just Newton's 3rd law for a charged particle in an external electromagnetic field. The antisymmetric tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7.15}$$

is called the *electromagnetic tensor*, and its six nonvanishing components are actually the $\vec{E}$ and $\vec{B}$ fields of Maxwell's equations.

We see that for $\mu = 1, 2, 3$

$$\frac{dp_k}{d\tau} = \frac{-dp^k}{d\tau} = \frac{-dp^k}{dt}\frac{dt}{d\tau} = Q\,u^0 F_{k0} + Q\,u^l F_{kl} \equiv Q\frac{dt}{d\tau}\left(F_{k0} + v^l F_{kl}\right)$$

or

$$\frac{dp^k}{dt} \equiv -Q\left(F_{k0} + v^l F_{kl}\right). \tag{7.16}$$

Eq. 7.16 has to agree with Newton's laws, with the force given by the Lorentz force

$$\vec{f} = Q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

so we can identify $-F_{k0}$ with $\vec{E}$. Is this correct? We see that

$$-F_{k0} = -\nabla^k A^0 + \frac{d}{dt}\left(\eta_{kk} A^k\right) = -\nabla^k A^0 - \frac{d}{dt}A^k \tag{7.17}$$

so indeed it is E^k.

What about $-Q\,v^l F_{kl}$? Consider the $k=3$ component for definiteness:

$$v^1 F_{31} + v^2 F_{32} + v^3 F_{33} \equiv v^1 F_{31} + v^2 F_{32}\,.$$

But $F_{31} = -\partial_z A_x + \partial_x A_z = -B^2$ (we have used $A^1 \equiv A_x$, *etc.* to make contact wth the Cartesian 3-dimensional notation); and similarly, $F_{32} = B^1$, so we easily see

$$-Q\,v^j F_{3j} = Q\left(v^1 B^2 - v^2 B^1\right) = Q\,(\vec{v}\times\vec{B}\,)_z\,.$$

Thus the space components of Eq. 7.14 give Newton's 3rd law, with the force given by the Lorentz force. What about the time component?

$$\frac{dp^0}{dt} = Q\,v^k F_{0k} = Q\,\vec{v}\cdot\vec{E}\,. \tag{7.18}$$

This is just the statement that

$$dp^0 = d\vec{x}\cdot\vec{f} \tag{7.19}$$

which we already know is true from the relation between work and energy.