Variational methods in mechanics and E&M

Electrodynamics in Minkowski space

Recall we found the equation of motion of a particle in a Lorentz vector field

$$\frac{dp_\mu}{d\tau} = QU^\nu F_{\mu\nu} \tag{8.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8.2}$$

is called the electromagnetic tensor, and its 6 components are actually the $\vec{E}$ and $\vec{B}$ fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

The interaction term in the Lagrangian (from which we derived Eq. 8.1) was

$$L_{int} = -Q\, U^\mu A_\mu (\vec{x}(t), t) \tag{8.3}$$

i.e. the vector field A_μ is evaluated at the instantaneous position of the particle.

Eq. 8.3 can be rewritten as

$$L_{int} = -\int d^3x\, J^\mu\, (\vec{x}, t)\, A_\mu (\vec{x}, t) \tag{8.4}$$

where

$$J^\mu(x) = \int d\tau\, Q\, \delta^{(4)} (x - \xi(\tau)) \frac{d\xi^\mu}{d\tau} \tag{8.5}$$

is the electromagnetic current density. Clearly, the current density for a collection of point particles is just

$$J^\mu(x) = \int d\tau \sum_n Q_n\, \delta^{(4)} (x - \xi_n(\tau)) \frac{d\xi_n^\mu}{d\tau}. \tag{8.6}$$

From its very form, $\partial_\mu J^\mu = 0$.

Now, suppose we want to derive Maxwell's equations of the electromagnetic field (displayed at the right) from an action principle: first we must write them in Lorentz covariant form.

Maxwell's Equations

$$\nabla \cdot \vec{E} = 4pr$$

$$\nabla \times \vec{B} = \tfrac{4p}{c} \vec{J} + \tfrac{1}{c} \vec{E}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \tfrac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

From

$$E^k = F_{0k} = -F^{0k}$$
$$B^j = -F_{kl} = -F^{kl} \tag{8.7}$$

(j, k, l are cyclic permutations of 1,2,3), we recover the Lorentz force

$$f^k = \frac{dp^k}{dt} = \frac{d\tau}{dt}(Q\, U_\nu\, F^{k\nu}) = -Q\, F^{0k} - Q\sum_{l=1}^{3} u^l F^{kl} = Q\, E^k + Q\,[\vec{u}\times\vec{B}]^k .$$

The first two (the pair with sources) of Maxwell's equations can then be written

$$\partial_0 F^{00} + \partial_k F^{k0} = \nabla\cdot\vec{E} = 4\pi\rho = 4\pi J^0$$

$$\partial_0 F^{0k} + \partial_l F^{lk} = -\frac{\partial E^k}{\partial t} + \varepsilon^{klj}\,\partial_l B^j = 4\pi J^k \tag{8.8}$$

$$\therefore\ \partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

Eq. 8.8 has the *form* of a Lorentz covariant equation, since ∂_μ and J^ν are both 4-vectors under Lorentz transformation.

The second (homogeneous) pair of Maxwell's equations can be written

$$\varepsilon^{\mu\nu\sigma\lambda}\,\partial_\nu F_{\sigma\lambda} = 0 \tag{8.9}$$

where $\varepsilon^{\mu\nu\sigma\lambda}$ is the totally antisymmetric (with respect to any pair of indices) tensor, defined so that $\varepsilon^{0123} = 1$. Non-zero elements are ± 1, obviously.

Clearly Eq. 8.9 and Eq. 8.8 are covariant *iff* $F_{\mu\nu}$ is a tensor. Is it? Anyone?

To show $F_{\mu\nu}$ is a tensor, it is enough to show A_μ is a vector. How do we do it? Go back to Maxwell's equations and let

$$\vec{B} = \nabla\times\vec{A}$$
$$\vec{E} = -\nabla A^0 - \partial_t \vec{A} \tag{8.10}$$

Then the last two Maxwell's equations are automatically satisfied, and the first two give

$$\partial_\mu \partial^\mu A^\nu = 4\pi J^\nu + \partial^\nu \Lambda \tag{8.11}$$

where

$$\Lambda = \partial_\mu A^\mu = \partial_t A^0 + \nabla\cdot\vec{A} . \tag{8.12}$$

Clearly we can always add some $\nabla\widetilde{\Lambda}$ to $\vec{A}$ because this can't change $\vec{B}$; and we can then add $-\partial_t\widetilde{\Lambda}$ to $\vec{E}$ because then $\vec{E}$ doesn't change. The result is called a *gauge transformation*. Then

$$\Lambda \to \Lambda - \partial_\mu \partial^\mu \widetilde{\Lambda} .$$

Since $\widetilde{\Lambda}$ is clearly a scalar, we can always choose $\Lambda = 0$ (if it isn't 0, find an appropriate $\widetilde{\Lambda}$ that makes it so). If we do this, the choice is manifestly Lorentz invariant and so

$$\partial_\mu \partial^\mu A^\nu = 4\pi J^\nu . \tag{8.13}$$

But since J^μ is a 4-vector, A^μ must also be one. Hence $F^{\mu\nu}$ is a tensor. QED.

Principle of Least Action

To have a Lorentz invariant action, we must write

$$\mathsf{A} = \int d^4x\, \mathsf{L}\left(A^\mu, \partial_\nu A^\mu\right) \tag{8.14}$$

where L is a scalar under Lorentz transformation. It has to be (at least) quadratic in A_μ and have no more than first derivatives of A^μ, in order to give Maxwell's equations when varied.

The possibilities are

$$F^{\mu\nu} F_{\mu\nu}$$

$$A^\mu A_\mu$$

$$\left(\partial_\mu A^\mu\right)^2$$

$$\ldots$$

Only $F^{\mu\nu} F_{\mu\nu}$is gauge invariant, hence it is the only possible term†. In the homework problems we saw that

$$F^{\mu\nu} F_{\mu\nu} = 2(\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E}) . \tag{8.15}$$

That is,

$$\mathsf{L} = \text{const} \times F^{\mu\nu} F_{\mu\nu} .$$

What is the constant? We now figure this out. The energy of a system can be derived from the Lagrangian by the transformation

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L . \tag{8.16}$$

H is called the ***Hamiltonian***. The analog for deriving Hamiltonian density H from a Lagrangian density is

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† …actually, the term $\varepsilon^{\mu\nu\sigma\lambda} F_{\mu\nu} F_{\sigma\lambda}$ is gauge-invariant, but it has odd parity under reflections. Since the electromagnetic interaction conserves parity, such a term would have to appear to the second power, but this would lead to a nonlinear electromagnetic theory, for which we have no experimental evidence at the macroscopic level.

$$\mathsf{H} = \dot{q}(x)\frac{\partial \mathsf{L}}{\partial \dot{q}(x)} - \mathsf{L}\,, \tag{8.17}$$

where, of course, if there is more than one field $q(x)$, we sum over all. Now specialize to EM fields in vacuum---in the absence of sources we can choose $A^0 = 0$, so find

$$\mathsf{L} = 2 \times \text{const} \times \left[\left(\nabla \times \vec{A}\right)^2 - \left(\partial_t \vec{A}\right)^2 \right] \tag{8.18}$$

or

$$\mathsf{H} = -2 \times \text{const} \times \left[(\vec{B})^2 + (\vec{E})^2 \right]. \tag{8.19}$$

But we also know, from integrating the work done moving charges in electric and magnetic fields, that the energy density of the electromagnetic field is

$$\mathsf{U} = \frac{1}{8\pi}\left[(\vec{B})^2 + (\vec{E})^2 \right], \tag{8.20}$$

hence

$$\text{const} = \frac{-1}{16\pi}\,,$$

$$\mathsf{L}_{EM} = \frac{-1}{16\pi}(F^{\mu\nu}F_{\mu\nu})\ .$$

The Euler-Lagrange equations for the electromagnetic field are thus (by an easy generalization from the particle case)

$$\partial_\mu\left(\frac{\partial \mathsf{L}}{\partial A_{\nu,\mu}}\right) - \frac{\partial \mathsf{L}}{\partial A_\nu} = 0\,; \tag{8.21}$$

taking the sum of pure-field and interaction Lagrangians to be

$$\mathsf{L} = \frac{-1}{16\pi}(F^{\mu\nu}F_{\mu\nu}) - J^\nu A_\nu\,, \tag{8.22}$$

we find, as before,

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu\,. \tag{8.23}$$