Local fields vs. action at a distance

Let us derive the vector potential of a moving charge†. We choose a gauge with $\partial_\mu A^\mu = 0$, so the first two Maxwell equations have the form

$$\partial_\kappa \partial^\kappa A^\mu = 4\pi J^\mu \tag{9.1}$$

Fourier transform both sides with respect to $\vec{x}$ and t, letting $k^\mu = (\omega, \vec{k})$:

$$\begin{Bmatrix} A^\mu(\vec{x}, t) \\ J^\mu(\vec{x}, t) \end{Bmatrix} = \frac{1}{(2\pi)^4} \int d^3k \int_{-\infty}^{+\infty} d\omega\, e^{ik \cdot x} \begin{Bmatrix} \widetilde{A}^\mu(\vec{k}, \omega) \\ \widetilde{J}^\mu(\vec{k}, \omega) \end{Bmatrix} \tag{9.2}$$

The gauge condition becomes $k_\mu \widetilde{A}^\mu = 0$, so (from Eq. 9.1)

$$-k \cdot k\, \widetilde{A}^\mu = 4\pi \widetilde{J}^\mu \tag{9.3}$$

Solving for $\widetilde{A}^\mu$ and substituting back into Eq. 9.2,

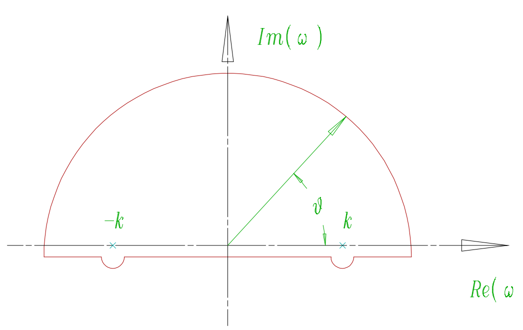
$$A^\mu(x) = \frac{1}{(2\pi)^4} \int d^3k \int_{-\infty}^{+\infty} d\omega\, e^{ik \cdot x} \frac{4\pi \widetilde{J}^\mu(\vec{k}, \omega)}{\vec{k} \cdot \vec{k} - \omega^2}, \tag{9.4}$$

$$= \int d^4x' \left[\frac{1}{(2\pi)^4} \int d^3k \int_{-\infty}^{+\infty} d\omega \frac{e^{ik \cdot (x - x')}}{\vec{k} \cdot \vec{k} - \omega^2} \right] 4\pi J^\mu(x') \;.$$

To calculate the vector potential we obviously need to evaluate the Green's function‡

$$D(x - x') = \left[\frac{1}{(2\pi)^4} \int d^3k \int_{-\infty}^{+\infty} d\omega \frac{e^{ik \cdot (x - x')}}{\vec{k} \cdot \vec{k} - \omega^2} \right] \tag{9.5}$$

The standard method for doing this is contour integration. Consider the integration contour shown to the right:



We will do the ω integral first. We are interested in the particular solution of the partial differential equation that gives a *retarded* EM potential A^μ. That is, causality requires that there be no vector potential in the region outside the light cone described by

† This is sometimes called the *Lienard-Weichart* potential.

‡ …in quantum field theory this is also called a *photon propagator*.

$$|\vec{x} - \vec{x}'| > |t - t'|,$$

i.e. a signal cannot propagate faster than light.

Now in the integral

$$I = \int_{-\infty}^{+\infty} d\omega \, \frac{e^{i\omega(t-t')}}{k^2 - \omega^2}, \qquad k = |\vec{k}|$$

there are obviously two places we have to be careful, indicated by × in the Figure: $\omega = \pm k$. These are *singularities*---in the language of complex variable theory they are classified as *simple poles*. As shown, we avoid them by detouring into the complex ω plane. Precisely how we choose the detour(s) determines whether we get retarded, advanced or mixed solutions to the original differential equation. Since here we want retarded solutions, we note that since

$$R = |\vec{x} - \vec{x}'| > 0,$$

if $\tau = t - t'$ is positive we want the integral to vanish for $R > \tau$ and to give a nonzero result in the opposite case. That is, if τ is positive, it refers to effects (at time t) that happen later than the cause (at time t'). With $\tau > 0$ we must close the contour in the upper half of the complex ω-plane.

Why is this? On the large (upper) semi-circle $\omega = \Omega\, e^{i\theta}$ the factor $e^{i\omega\tau}$ is dominated by $e^{-\Omega\,\tau\,\sin\theta}$ and the integrand therefore goes to 0 at least as fast as $1/\Omega^2$.

Conversely, for $\tau < 0$ we must close in the lower half of the complex w-plane, and here the integral must give exactly 0 in order to obey causality.

We must therefore arrange the singularities of the integral so they are included in the contour when $\tau > 0$, but are excluded when $\tau < 0$. They must be displaced upward from the $\mathrm{Re}(\omega)$ axis, or equivalently, we detour around them on the tiny semicircles shown in the drawing.

By Cauchy's theorem, since the singularities are simple poles, we get

$$I = \int_{-\infty}^{+\infty} d\omega \, \frac{e^{i\omega\tau}}{k^2 - \omega^2} = -2\pi i \left[\frac{e^{ik\tau}}{2k} - \frac{e^{-ik\tau}}{2k} \right] = \frac{2\pi}{k} \sin(k\tau)\, \theta(\tau) \quad . \tag{9.6}$$

Now we must do the integral over $\vec{k}$: First the angular integration gives

$$\int d\hat{k}\, e^{-i\vec{k}\cdot\vec{R}} f(|\vec{k}|) = 4\pi f(|\vec{k}|)\, \frac{\sin(kR)}{kR};$$

putting it all together, we find

$$D(x - x') = \frac{\theta(\tau)}{2\pi^2 R} \int_0^{\infty} dk \sin(kR) \sin(k\tau) \equiv \frac{\theta(\tau)}{8\pi^2 R} \int_{-\infty}^{+\infty} dk \left[\cos\Big(k(R-\tau)\Big) - \cos\Big(k(R+\tau)\Big) \right]$$

$$= \frac{\theta(\tau)}{4\pi R} \Big[\delta(R-\tau) - \delta(R+\tau) \Big] \equiv \frac{\theta(\tau)}{4\pi R}\, \delta(R-\tau) \quad . \tag{9.7}$$

Thus

$$A^{\mu}(x) = \int d^3x' \int_{-\infty}^{t} dt' \frac{J^{\mu}(\vec{x}', t')}{|\vec{x}' - \vec{x}|} \delta(|\vec{x}' - \vec{x}| - t + t')$$

$$= \int d^3x' \frac{J^{\mu}(\vec{x}', t - |\vec{x}' - \vec{x}|)}{|\vec{x}' - \vec{x}|} \tag{9.8}$$

Now consider a point charge

$$J^{\mu}(\vec{x}, t) = Q \frac{d\xi^{\mu}(t)}{dt} \delta^{(3)}\left(\vec{x} - \vec{\xi}(t)\right) \tag{9.9}$$

Then

$$A^{\mu}(x) = Q \int_{-\infty}^{t} dt' \frac{1}{|\vec{x} - \vec{\xi}(t')|} \frac{d\xi^{\mu}}{dt'} \delta(t' - t + |\vec{x} - \vec{\xi}(t')|)$$

and since, if $f(t)|_{t=t_0} = 0$,

$$\delta(f(t)) = \delta\left((t-t_0) \left.\frac{df}{dt}\right|_{t_0}\right) \equiv \frac{1}{|\dot{f}(t_0)|} \delta(t-t_0) \tag{9.10}$$

we have

$$A^{\mu}(x) = Q \frac{d\xi^{\mu}}{dt'} \left| |\vec{x} - \vec{\xi}(t')| - \frac{d\vec{\xi}}{dt'} \cdot (\vec{x} - \vec{\xi}(t')) \right|^{-1} \tag{9.11}$$

where

$$t' = t - |\vec{x} - \vec{\xi}(t')| \tag{9.12}$$

is the "retarded time". The *Lienard-Weichart potential*, Eq. 9.11, can be thought of as action-at-a-distance because if the particle has a constant velocity $\vec{u}$, then

$$t' = t - |\vec{x} - \vec{\xi}_0 - \vec{u}\, t'| .$$

It is then easy to see that the Coulomb force is directed along the vector from the point of observation to the particle's present position. For an accelerated source, the force is directed to where the particle *would have been* if it had continued on without acceleration.

Thus it might seem plausible to formulate force laws without positing local fields to mediate them, as some of Newton's successors did.

As Ohanian and Ruffini point out (§2.7), momentum is not conserved by action-at-a-distance forces. Action and reaction are not balanced for systems accelerated by their mutual forces. The only place the missing momentum can be hiding is in the fields themselves.