Linear field approximation to gravitation

Masslessness of photon (a digression with some physical interest)

We have seen already that the gauge invariance of electromagnetism is related to the absence of a mass term in the Lagrangian†, of the form

$$\mathsf{L}_{mass} = \frac{1}{2} m^2 A^{\mu} A_{\mu} \qquad (10.1)$$

Equation 10.1 is not invariant under the gauge transformation

$$A^{\mu} \to A^{\mu} + \partial^{\mu} \Lambda ,$$

because it obviously develops additional terms proportional to Λ.

We also saw that conservation of electric charge, expressed *via*

$$\partial_{\mu} J^{\mu} = 0$$

is connected with gauge invariance, since

$$\mathsf{L}_{int} = -J^{\mu} A_{\mu}$$

transforms into

$$\mathsf{L}_{int} = -J^{\mu} A_{\mu} - J^{\mu} \partial_{\mu} \Lambda \equiv -J^{\mu} A_{\mu} - \partial_{\mu} \left(J^{\mu} \Lambda \right) + \Lambda \, \partial_{\mu} J^{\mu} .$$

The last term vanishes because of charge conservation, hence the change in the Lagrangian density is a pure divergence, which cannot contribute to the Euler-Lagrange equations. Thus, for esthetic reasons we believe the photon is massless.

However, physics----as opposed to philosophy----is an experimental science. What does experiment say? A massive photon would lead to a modified Coulomb potential $Q \frac{e^{-mr}}{r}$ (in units with $\hbar = c = 1$). The current best limit‡ on the photon mass, m_{γ} is $m_{\gamma} \leq 6 \times 10^{-16}$ eV/c^2, arising from the detection and mapping of the magnetic field of the planet Jupiter.

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† We also saw why Eq. 10.1 is called a *mass* term.

‡ See *PRL* **35** (1975) 1402.

Why does gravitation couple to $T^{\mu\nu}$?

1. The Principle of Equivalence, as determined experimentally by Galileo and by the Eötvos experiment ⇒ that gravitation couples to energy; consider a hot and a cold object of the same composition and size. Which is heavier? Obviously the hot one. Why? Near the surface of the Earth, the weight is given by

$$W = g\sum_{n=1}^{N} \frac{m_n}{\sqrt{1-u_n^2}} \approx gM + g\left(\frac{3}{2}Nk_BT\right) \tag{10.2}$$

2. If gravitation were a vector, then the force would couple to the “charge” (mass) as in electromagnetism,

$$\vec{F} = -\nabla A^0$$

hence with $A^0 = gz$,

$$W = gM \tag{10.3}$$

independent of T.

3. If gravitation were a scalar field, then we would vary

$$L = -\left(mc^2 + S\right)\sqrt{1-\vec{u}\cdot\vec{u}/c^2}$$

to find

$$\vec{F} = -\nabla S\sqrt{1-\vec{u}\cdot\vec{u}/c^2}\ .$$

That is, the weight would decrease with temperature:

$$W \approx mg - g\left(\frac{3}{2}Nk_BT\right). \tag{10.4}$$

Also, we would find that if the field were a scalar the source would have to be a scalar also, because the field equation would have to be the (Lorentz-invariant) generalization of Newton’s Law of Universal Gravitation,

$$\partial_\mu\partial^\mu\varphi = 4\pi\sigma$$

hence σ would be $\propto T^\mu{}_\mu$. But for the electromagnetic field, $T^\mu{}_\mu = 0$. That is, the electromagnetic contribution to the mass-energy of a body could not contribute to its gravitational mass.

4. A tensor gravitational field would agree with the Principle of equivalence: as we saw in the homework solutions, the Lagrangian for a slowly moving body,

$$L \approx -mc^2\sqrt{1-\vec{u}\cdot\vec{u}/c^2} - m\frac{h^{00}(r)}{\sqrt{1-\vec{u}\cdot\vec{u}/c^2}}$$

predicts a gravitational force proportional to the energy content,

$$\vec{F} = -\nabla\left(m\frac{h^{00}(r)}{\sqrt{1-\vec{u}\cdot\vec{u}/c^2}}\right).$$

We therefore conclude the gravitational field must be represented by a second-rank tensor, $h^{\mu\nu}$. We therefore seek a field equation like

$$\varphi^{\mu\nu} = -4\pi G\, T^{\mu\nu}\,, \qquad (10.5)$$

where $\varphi^{\mu\nu}$ is a tensor constructed from the (tensor) gravitational potential $h^{\mu\nu}$ by the usual operations of differentiation and/or multiplication by the Minkowski tensor $\eta^{\mu\nu}$.

The source term in the field equation must also be a rank-2 tensor, whose only reasonable candidate is the energy-momentum tensor, $T^{\mu\nu}$.

Clearly, $\varphi^{\mu\nu}$ must satisfy

$$\varphi^{\mu\nu} = \varphi^{\nu\mu} \qquad (10.6)$$

and

$$\partial_\mu\, \varphi^{\mu\nu} = 0\,. \qquad (10.7)$$

Equation 10.7 follows because

$$\partial_\mu\, T^{\mu\nu} = 0\,.$$

Now how can we construct $\varphi^{\mu\nu}$? Suppose we start with $h^{\mu\nu}$ (which we may obviously assume symmetric); what kinds of terms can we make out of $h^{\mu\nu}$ under the restrictions:

- $\varphi^{\mu\nu}$ must be linear in $h^{\mu\nu}$;
- $\varphi^{\mu\nu}$ can involve derivatives no higher than second-order.

Let

$$\varphi^{\mu\nu} = m^2\, h^{\mu\nu} + m'^{\,2}\, \eta^{\mu\nu}\, h^{\kappa}{}_{\kappa} + (1)\, \partial^\kappa\partial_\kappa\, h^{\mu\nu} + b\left[\partial^\mu\partial_\kappa\, h^{\kappa\nu} + \partial^\nu\partial_\kappa\, h^{\mu\kappa}\right] +$$
$$+\, c\, \partial^\mu\partial^\nu\, h^{\kappa}{}_{\kappa} + d\, \eta^{\mu\nu}\, \partial^\kappa\partial_\kappa\, h^{\lambda}{}_{\lambda} + e\, \eta^{\mu\nu}\, \partial_\kappa\partial_\lambda\, h^{\kappa\lambda} \qquad (10.8)$$

We have chosen the coefficient of $\partial^\kappa\partial_\kappa\, h^{\mu\nu}$ to be unity to set the overall scale of $h^{\mu\nu}$.

Since gravitation is observed to act at least over distances of order of the radius of globular clusters, we may surmise it is a long range force and that the mass terms are negligible:

$$m = m' = 0\,.$$

Then from Eq. 10.7 we have

$$\partial_\mu\, \varphi^{\mu\nu} = 0 = \partial^\kappa\partial_\kappa\, \partial_\mu h^{\mu\nu}\, (l+b) + (b+e)\, \partial^\nu\partial_\kappa\partial_\lambda\, h^{\kappa\lambda} + (c+d)\, \partial^\nu\, \partial^\kappa\partial_\kappa\, h^{\lambda}{}_{\lambda} \qquad (10.9)$$

Hence

$$\begin{aligned} 1+b &= 0 \\ b+e &= 0 \\ c+d &= 0\,. \end{aligned} \qquad (10.10)$$

Thus

$$\varphi^{\mu\nu} = \partial^\kappa\partial_\kappa h^{\mu\nu} - \left(\partial^\mu\partial_\kappa h^{\kappa\nu} + \partial^\nu\partial_\kappa h^{\mu\kappa}\right) + \eta^{\mu\nu}\partial_\kappa\partial_\lambda h^{\kappa\lambda} + $$
$$+ c\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\kappa\partial_\kappa\right)h^\lambda{}_\lambda \ . \qquad (10.11)$$

The only free parameter—after linearity in $h^{\mu\nu}$ and the conservation of $T^{\mu\nu}$ are required (a form of gauge condition)—is c. The question we must answer next is, "What does c stand for?" Can we choose c arbitrarily or is it physical ?

Consider the "gauge" transformation

$$h^{\mu\nu} \to \tilde{h}^{\mu\nu} - C\,\eta^{\mu\nu} h \qquad (10.12)$$

where

$$h \overset{df}{=} h^\lambda{}_\lambda \ ,$$

and similarly with $\tilde{h}$. Since $\eta^{\mu\nu}\eta_{\mu\nu} = 4$, we see

$$h \to \tilde{h}\,(1 - 4C) \ ; \qquad (10.13)$$

so that

$$\tilde{\varphi}^{\mu\nu} = \partial^\kappa\partial_\kappa \tilde{h}^{\mu\nu} - \left(\partial^\mu\partial_\kappa \tilde{h}^{\kappa\nu} + \partial^\nu\partial_\kappa \tilde{h}^{\mu\kappa}\right) + \eta^{\mu\nu}\partial_\kappa\partial_\lambda \tilde{h}^{\kappa\lambda} + $$
$$+ \left[2C + c\,(1 - 4C)\right]\left[\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\kappa\partial_\kappa\right]\tilde{h} \ . \qquad (10.14)$$

Writing

$$\tilde{c} = 2C + c\,(1 - 4C) \ ,$$

we see that $\tilde{\varphi}^{\mu\nu}$ is the same function of $\tilde{h}^{\mu\nu}$ as $\varphi^{\mu\nu}$ is of $h^{\mu\nu}$, except with c replaced by $\tilde{c}$. Obviously we can make $\tilde{c}$ anything we want it to be. For example, choosing

$$C = \frac{c}{2(2c - 1)} \ ,$$

we can make $\tilde{c} = 0$ and simply drop this term; alternatively, we could let

$$C = \frac{c - 1}{2(2c - 1)} \ ,$$

making $\tilde{c} = 1$. We choose the latter, obtaining the linearized, Lorentz-invariant gravitational field equations

$$\partial^\kappa\partial_\kappa h^{\mu\nu} - \left(\partial^\mu\partial_\kappa h^{\kappa\nu} + \partial^\nu\partial_\kappa h^{\mu\kappa}\right) + \eta^{\mu\nu}\partial_\kappa\partial_\lambda h^{\kappa\lambda} + $$
$$+ \left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\kappa\partial_\kappa\right)h^\lambda{}_\lambda = -4\pi T^{\mu\nu} \qquad (10.15)$$

Equation 10.15 is invariant under the gauge transformation

$$h^{\mu\nu} \to h^{\mu\nu} + \frac{1}{2}\left[\partial^\mu\Lambda^\nu + \partial^\nu\Lambda^\mu\right] = \tilde{h}^{\mu\nu} \ . \qquad (10.16)$$

We leave the proof as an exercise for the student.