Linear field approximation to gravitation II

Gravitational field of a distribution of matter

Recall that we had derived the field equation, by analogy with electromagnetism,

$$\partial^\kappa \partial_\kappa h^{\mu\nu} - \left(\partial^\mu \partial_\kappa h^{\kappa\nu} + \partial^\nu \partial_\kappa h^{\mu\kappa}\right) + \eta^{\mu\nu} \partial_\kappa \partial_\lambda h^{\kappa\lambda} + \\ + \left(\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^\kappa \partial_\kappa\right) h^\lambda{}_\lambda = -KT^{\mu\nu} \tag{10.15}$$

Eq. 10.15 is invariant under the gauge transformation

$$h^{\mu\nu} \to h^{\mu\nu} + \frac{1}{2}\left[\partial^\mu \Lambda^\nu + \partial^\nu \Lambda^\mu\right] = \widetilde{h}^{\mu\nu} \tag{10.16}$$

Assume the gauge condition

$$\partial_\mu \left(h^{\mu\nu} - \tfrac{1}{2} \eta^{\mu\nu} h\right) = 0 \tag{11.1}$$

(we can always pick a gauge function $\Lambda(x)$ such that this is so).

Then the field equ'ns become

$$\partial^\kappa \partial_\kappa \left(h^{\mu\nu} - \tfrac{1}{2} \eta^{\mu\nu} h\right) = -KT^{\mu\nu} . \tag{11.2}$$

Let

$$\zeta^{\mu\nu} \overset{df}{=} h^{\mu\nu} - \tfrac{1}{2} \eta^{\mu\nu} h$$

so that

$$\zeta = h - \tfrac{1}{2} \times 4 \times h = -h$$

$$\zeta^{\mu\nu} = h^{\mu\nu} + \tfrac{1}{2} \eta^{\mu\nu} \zeta$$

$$h^{\mu\nu} = \zeta^{\mu\nu} - \tfrac{1}{2} \eta^{\mu\nu} \zeta .$$

It is much easier to calculate ζ from

$$\partial^\kappa \partial_\kappa \zeta^{\mu\nu} = -KT^{\mu\nu} \tag{11.3}$$

than $h^{\mu\nu}$ from Eq. 11.2.

Example

We shall now calculate the gravitational field of a point mass. The energy-momentum tensor of a point particle at rest is

$$T^{\mu\nu} = \begin{pmatrix} M\delta^{(3)}(\vec{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11.4}$$

giving

$$-\nabla^2 \zeta^{00}(\vec{x}) = -KM\,\delta^{(3)}(\vec{x}) \tag{11.5}$$

so

$$\zeta^{00}(\vec{x}) = -\frac{KM}{4\pi|\vec{x}|}\,. \tag{11.6}$$

We see that $\zeta = \zeta^{00}$, so that $h^{00} = \frac{1}{2}\zeta^{00}$.

Equation of motion of a test particle

Newton's 2nd Law for a test particle† of mass m in the above field is

$$\frac{d\vec{p}}{dt} = -\nabla\left(\frac{-GMm}{|\vec{x}|}\right) \tag{11.7}$$

or

$$\frac{d}{dt}\left(m\frac{\vec{u}}{\sqrt{1-\vec{u}\cdot\vec{u}}}\right) = -m\frac{4\pi G}{K}\nabla\zeta^{00} = -m\frac{8\pi G}{K}\nabla h^{00}$$

which could be expressed as

$$\delta\int dt\, L(\vec{x}(t), \vec{u}(t)) = 0$$

$$L = \underbrace{-m\sqrt{1-\vec{u}\cdot\vec{u}}}_{\text{Scalar}}\ \underbrace{-m\,h^{00}\frac{8\pi G}{K}}_{\text{Tensor}}\,.$$

where

This is no good! The Lagrangian (times dt) is suppose to be a Lorentz scalar. How can we make the h^{00} term into a scalar?

Clearly the right way to do this is

$$h^{00}\,dt \to h^{\mu\nu}U_\mu U_\nu\,d\tau\,. \tag{11.8}$$

It will then be convenient to rewrite tha action as

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† A "test particle" is one whose mass is so small we may neglect its contribution to the graviational field.

$$\mathsf{A} \rightarrow -\int d\tau \left(\frac{1}{2} m \, \eta^{\mu\nu} + m \frac{16\pi G}{2K} h^{\mu\nu} \right) U_\mu U_\nu \, . \tag{11.9}$$

If we choose $K = 16\pi G$ and call

$$\eta^{\mu\nu} + h^{\mu\nu} \overset{df}{=} g^{\mu\nu},$$

we see that

$$L = -\frac{1}{2} m \left(\eta^{\mu\nu} + \frac{16\pi G}{K} h^{\mu\nu} \right) U_\mu U_\nu$$

has the form of a metric in a curved space. This is one way we can recognize that gravitation can be identified with geometry.

Why gravitation ⇔ geometry

The Principle of Equivalence says that it is impossible to distinguish gravitational effects from accelerations. Consider a rotating disk. According to Special Relativity, its circumference (as measured by a stationary observer) will be ($g = R\omega^2$)

$$2\pi R \sqrt{1 - (R\omega)^2/c^2} = 2\pi R \sqrt{1 - gR/c^2} \, .$$

However, the radius is always perpendicular to the velocity, hence is the same in the stationary system as in the rest frame of the disk. In consequence, the geometrical constant π' measured in an accelerated frame must differ from π in an unaccelerated frame:

$$\pi' = \pi \sqrt{1 - gR/c^2} \, .$$

If we express the effect in terms of the centrifugal potential energy per unit mass,

$$\varphi = \frac{1}{2} (R\omega)^2$$

we have

$$\pi' = \pi \sqrt{1 - 2\varphi/c^2} \, . \tag{11.10}$$

That is, a gravitational potential affects the geometry (because we cannot tell one kind of acceleration from another).

Relativistic motion in a gravitational field

We now consider the relativistic equation of motion of a test particle:

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial U_\mu} \right) - \frac{\partial L}{\partial \xi_\mu} = 0 \, . \tag{11.11}$$

Ignoring the factor $\frac{1}{2} m$,

$$\frac{d}{d\tau} \left(g^{\mu\nu} \, U_\nu \right) - \frac{1}{2} U_\kappa U_\nu \frac{\partial h^{\kappa\nu}}{\partial \xi_\mu} = 0 \, . \tag{11.12}$$

Now,

$$\frac{d}{d\tau}\left(g^{\mu\nu}\,U_\nu\right) = \frac{d}{d\tau}\,U^\mu \;+\; h^{\mu\nu}\,\frac{d}{d\tau}\,U_\nu \;+\; U_\kappa U_\nu\,\frac{\partial h^{\kappa\nu}}{\partial \xi_\mu} \tag{11.13}$$

so, to leading order (in gravitational problems, kinetic and potential energies are usually comparable, so $h^{\mu\nu}\,\frac{d}{d\tau}\,U_\nu$ is a correction of order ζ^2),

$$\frac{d}{d\tau}\,U^\mu \;+\; U_\kappa U_\nu\,\frac{\partial h^{\kappa\nu}}{\partial \xi_\kappa} - \frac{1}{2}\,U_\kappa U_\nu\,\frac{\partial h^{\kappa\nu}}{\partial \xi_\mu} = 0\,. \tag{11.14}$$

In the next lecture we shall look at some consequences of Eq. 11.14, both for particle motion and for scattering light by a gravitational field.