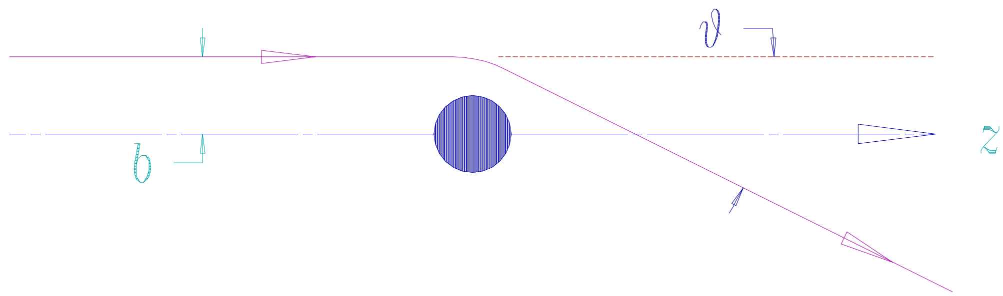
Linear field approximation to gravitation III

Deflection of a particle by gravitational field

We imagine a particle that is deflected by a small angle, owing to a near collision with a source of gravitation, as shown below.



Case 1: Newtonian gravitation

Assume the scattering is in the *x-z* plane. Newton's 2nd law gives (we assume the gravitational mass of the particle is its total energy ε).

$$\frac{dp^x}{dt} = F^x = \frac{-MG\varepsilon\, b}{\left(b^2 + z^2\right)^{3/2}} \tag{12.1}$$

which we integrate (using $dt \approx dz/v$) to get

$$\Delta p^x = \int_{-\infty}^{+\infty} F^x\, dt \approx \frac{-MG\varepsilon}{v}\int_{-\infty}^{+\infty} \frac{b\, dz}{\left(b^2 + z^2\right)^{3/2}}. \tag{12.2}$$

The angular deflection is (for small angles)

$$\Delta\theta \approx \frac{\Delta p^x}{|\vec{p}|} \approx \frac{-MG}{v^2\, b}\int_{-\infty}^{+\infty} d\zeta\,(1+\zeta^2)^{-3/2} = \frac{-2MG}{v^2\, b}. \tag{12.3}$$

Remember Eq. 12.3---we shall return to it.

Thus, the Newtonian theory predicts a net angular deflection of a massless particle (v=c) by

$$\Delta\theta \approx \frac{-2MG}{c^2\, b}. \tag{12.4}$$

Case 2: General-relativistic prediction

We saw in Lecture 11 that the field of a static source with mass M is

$$h^{00} = \tfrac{1}{2}\varphi^{00} = \frac{-2MG}{r}$$
$$h^{kk} = -\tfrac{1}{2}\eta^{kk}\varphi = h^{00} \tag{12.5}$$

We found the equation of motion of a test particle to be

$$\frac{d}{d\tau}U^{\mu} + h^{\mu\nu}\frac{d}{d\tau}U_{\nu} + U_{\kappa}U_{\nu}\frac{\partial h^{\mu\nu}}{\partial\xi_{\kappa}} - \tfrac{1}{2}U_{\kappa}U_{\nu}\frac{\partial h^{\kappa\nu}}{\partial\xi_{\mu}} = 0 \tag{12.6}$$

and we agreed to neglect $h^{\mu\nu}\frac{d}{d\tau}U_{\nu}$ as being of order $(h)^2$---which we are already neglecting in making a linear approximation. Thus to this order,

$$\frac{d}{d\tau}U^{\mu} + U_{\kappa}U_{\nu}\frac{\partial h^{\mu\nu}}{\partial\xi_{\kappa}} - \tfrac{1}{2}U_{\kappa}U_{\nu}\frac{\partial h^{\kappa\nu}}{\partial\xi_{\mu}} = 0\,. \tag{12.7}$$

We now insert Eq. 12.5 into Eq. 12.7, component by component. Initially,

$$U^{\mu} = \begin{pmatrix}\gamma\\0\\0\\u\gamma\end{pmatrix}, \qquad U_{\mu} = \begin{pmatrix}\gamma\\0\\0\\-u\gamma\end{pmatrix} \tag{12.8}$$

where, as usual, $\gamma = \frac{1}{\sqrt{1-u^2}}$.

Thus,

$$\frac{d^2z}{d\tau^2} - \gamma^2 u^2\,\partial_z h + \tfrac{1}{2}\gamma^2\,\partial_z h + \tfrac{1}{2}\gamma^2 u^2\,\partial_z h = 0\,,$$

and similarly for $\frac{d^2t}{d\tau^2}$ and $\frac{d^2x}{d\tau^2}$; that is,

$$\frac{d^2z}{d\tau^2} + \tfrac{1}{2}\partial_z h = 0$$
$$\frac{d^2t}{d\tau^2} + u\gamma^2\,\partial_z h = 0 \tag{12.9}$$
$$\frac{d^2x}{d\tau^2} + \tfrac{1}{2}(1+u^2)\,\gamma^2\,\partial_z h = 0\,.$$

The third of equations 12.9 follows because U^x may be considered always small relative to U^z.

From Eq. 12.9 we see $U^z \approx$ const., hence we can integratethe equation for $\frac{d^2x}{d\tau^2}$ to get

$$\frac{du^x}{dt} + \frac{1}{2}(1+u^2)\,\gamma\,\partial_z h = 0\,, \tag{12.10}$$

or since $dt \approx dz/u$,

$$\Delta u^x \approx -\frac{1}{2}(1+u^2)\,\gamma\int_{-\infty}^{+\infty}\frac{2MGx\,dz}{u\left(x^2+z^2\right)^{3/2}}$$

which gives the angular deflection

$$\Delta\theta \approx \frac{\Delta u^x}{u} = -(1+u^{-2})\,\frac{2MG}{b}\,. \tag{12.11}$$

We see that for light-like particles,

$$\Delta\theta \approx \frac{-4MG}{b}\,, \tag{12.12}$$

which is twice the Newtonian prediction. It is interesting that Einstein first gave the Newtonian result (1911) and only later gave the correct result . The psychological impact would have been far less, had the measurement of the deflection of light rays by the Sun been carried out between 1911 and 1916, rather than in 1919.

Linear field approximation to gravitation II

Gravitational field of a distribution of matter

Recall that we had derived the field equation, by analogy with electromagnetism,

$$\partial^\kappa\partial_\kappa h^{\mu\nu} - \left(\partial^\mu\partial_\kappa h^{\kappa\nu} + \partial^\nu\partial_\kappa h^{\mu\kappa}\right) + \eta^{\mu\nu}\partial_\kappa\partial_\lambda h^{\kappa\lambda} + \\ + \left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\kappa\partial_\kappa\right)h^\lambda{}_\lambda = -KT^{\mu\nu} \tag{10.15}$$

Eq. 10.15 is invariant under the gauge transformation

$$h^{\mu\nu} \to h^{\mu\nu} + \frac{1}{2}\left[\partial^\mu\Lambda^\nu + \partial^\nu\Lambda^\mu\right] = \tilde{h}^{\mu\nu} \tag{10.16}$$

Assume the gauge condition

$$\partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = 0 \tag{11.1}$$

(we can always pick a gauge function $\Lambda(x)$ such that this is so).

Then the field equ'ns become

$$\partial^\kappa\partial_\kappa\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = -KT^{\mu\nu}. \tag{11.2}$$

Let

$$\zeta^{\mu\nu} \overset{df}{=} h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$$

so that

$$\zeta = h - \frac{1}{2}\times 4\times h = -h$$

$$\zeta^{\mu\nu} = h^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\zeta$$

$$h^{\mu\nu} = \zeta^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\zeta\ .$$

It is much easier to calculate ζ from

$$\partial^\kappa\partial_\kappa\zeta^{\mu\nu} = -KT^{\mu\nu} \tag{11.3}$$

than $h^{\mu\nu}$ from Eq. 11.2.

Example

We shall now calculate the gravitational field of a point mass. The energy-momentum tensor of a point particle at rest is

$$T^{\mu\nu} = \begin{pmatrix} M\delta^{(3)}(\vec{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11.4}$$

giving

$$-\nabla^2 \zeta^{00}(\vec{x}) = -KM\,\delta^{(3)}(\vec{x}) \tag{11.5}$$

so

$$\zeta^{00}(\vec{x}) = -\frac{KM}{4\pi|\vec{x}|}\,. \tag{11.6}$$

We see that $\zeta = \zeta^{00}$, so that $h^{00} = \frac{1}{2}\zeta^{00}$.

Equation of motion of a test particle

Newton's 2nd Law for a test particle† of mass m in the above field is

$$\frac{d\vec{p}}{dt} = -\nabla\left(\frac{-GMm}{|\vec{x}|}\right) \tag{11.7}$$

or

$$\frac{d}{dt}\left(m\frac{\vec{u}}{\sqrt{1-\vec{u}\cdot\vec{u}}}\right) = -m\,\frac{4\pi G}{K}\,\nabla\,\zeta^{00} = -m\,\frac{8\pi G}{K}\,\nabla\,h^{00}$$

which could be expressed as

$$\delta\int dt\,L(\vec{x}\,(t), \vec{u}\,(t)\,) = 0$$

where

This is no good! The Lagrangian (times dt) is suppose to be a Lorentz scalar. How can we make the h^{00} term into a scalar?

Clearly the right way to do this is

$$h^{00}\,dt \;\to\; h^{\mu\nu}\,U_\mu U_\nu\,d\tau\,. \tag{11.8}$$

It will then be convenient to rewrite tha action as

$$A \;\to\; -\int d\tau\left(\frac{1}{2}\,m\,\eta^{\mu\nu} + m\,\frac{16\pi G}{2K}\,h^{\mu\nu}\right)U_\mu U_\nu\,. \tag{11.9}$$

If we choose $K = 16\pi G$ and call

† A "test particle" is one whose mass is so small we may neglect its contribution to the graviational field.

$$\eta^{\mu\nu} + h^{\mu\nu} \overset{df}{=} g^{\mu\nu},$$

we see that

$$L = -\frac{1}{2} m \left(\eta^{\mu\nu} + \frac{16\pi G}{K} h^{\mu\nu}\right) U_\mu U_\nu$$

has the form of a metric in a curved space. This is one way we can recognize that gravitation can be identified with geometry.

Why gravitation ⇔ geometry

The Principle of Equivalence says that it is impossible to distinguish gravitational effects from accelerations. Consider a rotating disk. According to Special Relativity, its circumference (as measured by a stationary observer) will be ($g = R\omega^2$)

$$2\pi R \sqrt{1 - (R\omega)^2/c^2} = 2\pi R \sqrt{1 - gR/c^2} .$$

However, the radius is always perpendicular to the velocity, hence is the same in the stationary system as in the rest frame of the disk. In consequence, the geometrical constant π' measured in an accelerated frame must differ from π in an unaccelerated frame:

$$\pi' = \pi \sqrt{1 - gR/c^2} .$$

If we express the effect in terms of the centrifugal potential energy per unit mass,

$$\varphi = \frac{1}{2} (R\omega)^2$$

we have

$$\pi' = \pi \sqrt{1 - 2\varphi/c^2} . \tag{11.10}$$

That is, a gravitational potential affects the geometry (because we cannot tell one kind of acceleration from another).

Relativistic motion in a gravitational field

We now consider the relativistic equation of motion of a test particle:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\mu}\right) - \frac{\partial L}{\partial \xi_\mu} = 0 . \tag{11.11}$$

Ignoring the factor $\frac{1}{2} m$,

$$\frac{d}{d\tau}\left(g^{\mu\nu} U_\nu\right) - \frac{1}{2} U_\kappa U_\nu \frac{\partial h^{\kappa\nu}}{\partial \xi_\mu} = 0 . \tag{11.12}$$

Now,

$$\frac{d}{d\tau}\left(g^{\mu\nu} U_\nu\right) = \frac{d}{d\tau} U^\mu + h^{\mu\nu} \frac{d}{d\tau} U_\nu + U_\kappa U_\nu \frac{\partial h^{\kappa\nu}}{\partial \xi_\mu} \tag{11.13}$$

so, to leading order (in gravitational problems, kinetic and potential energies are usually comparable, so $h^{\mu\nu} \frac{d}{d\tau} U_\nu$ is a correction of order ζ^2),

$$\frac{d}{d\tau}U^{\mu} + U_{\kappa}U_{\nu}\frac{\partial h^{\kappa\nu}}{\partial \xi_{\kappa}} - \frac{1}{2}U_{\kappa}U_{\nu}\frac{\partial h^{\kappa\nu}}{\partial \xi_{\mu}} = 0\,. \tag{11.14}$$

In the next lecture we shall look at some consequences of Eq. 11.14, both for particle motion and for scattering light by a gravitational field.

$$L = \underbrace{-m\sqrt{1 - \vec{u}\cdot\vec{u}}}_{\text{Scalar}} \underbrace{-m\,h^{00}\,\frac{8\pi G}{K}}_{\text{Tensor}} .$$

Gravitation and Cosmology

Relativistic motion in a gravitational field