Linear field approximation, IV

Retardation of light in a gravitational field

We saw in §12 that the action for a particle in a weak field was

$$\mathsf{A} = \int d\tau\, \Lambda(\xi, \partial_\mu \xi) = \int d\tau \frac{1}{2} m \left(\eta^{\mu\nu} + h^{\mu\nu}\right) U_\mu U_\nu \tag{13.1}$$

where $U_\mu \overset{df}{=} \frac{d\xi_\mu}{d\tau}$, so the canonical momentum is

$$p^\mu \overset{df}{=} \frac{\partial \Lambda}{\partial U_\mu} = m\left(U^\mu + h^{\mu\nu} U_\nu\right) \tag{13.2}$$

and the equations of motion are

$$\frac{dp^\mu}{d\tau} = \frac{1}{2} m\, U_\kappa U_\nu \frac{\partial h^{\kappa\nu}}{\partial \xi_\mu} \approx \frac{1}{2} p_\kappa U_\nu \frac{\partial h^{\kappa\nu}}{\partial \xi_\mu}, \tag{13.3}$$

where eq. 13.3 follows from our agreement to ignore terms of higher order in h. Thus for a particle travelling at near-light speed (or at c, for that matter) and weakly deflected by a gravitational field, recalling that for a static source

$$h^{\mu\nu} = \begin{pmatrix} \frac{-2MG}{r} & 0 & 0 & 0 \\ 0 & \frac{-2MG}{r} & 0 & 0 \\ 0 & 0 & \frac{-2MG}{r} & 0 \\ 0 & 0 & 0 & \frac{-2MG}{r} \end{pmatrix}$$

we have

$$\begin{aligned} dp^z &= -\frac{1}{2} p^0\, dt \frac{\partial h^{00}}{\partial z} - \frac{1}{2} p^z\, dz \frac{\partial h^{00}}{\partial z} \\ dp^0 &= 0 \end{aligned} \tag{13.4}$$

(static source, $\partial_t h^{00} = 0$). Since $dt \approx dz$ for $v \approx c$, we have

$$dp^z \approx -\frac{\partial h^{00}}{\partial z} p^0\, dz \tag{13.5}$$

We may integrate w.r.t. z to get

$$p^z(z) \approx p^z(\pm\infty) - p^0 h^{00}(r)\,. \tag{13.6}$$

The momentum at $z = \pm\infty$ is $\approx p^0$, hence we may say the group velocity of the particle at position z is

$$v(z) \overset{df}{=} \frac{dp^0}{dp^z} = \left(1 + \frac{2MG}{r}\right)^{-1}. \tag{13.7}$$

The time delay in passing the object at impact parameter b is then given by the integral of the time to go a distance dz, relative to what it would have been with no source of gravitation:

$$\Delta t = \int_{Z_1}^{Z_2} \left(\frac{dz}{v(b,z)} - \frac{dz}{1} \right). \qquad (13.8)$$

The integral 13.8 can be performed in closed form, giving!†•

$$\Delta t = \frac{2M_\odot G}{c^3} \left[\sinh^{-1}\left(Z_2/b\right) - \sinh^{-1}\left(Z_1/b\right) \right].$$

If $|Z_{1,2}| \gg b$, then the above expression simplifies to

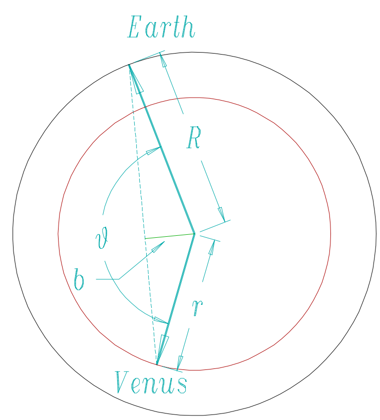
$$\Delta t \approx \frac{2M_\odot G}{c^3} \log\left(\frac{4Z_2\,|Z_1|}{b^2} \right).$$

With Earth and Venus shown as in the drawing to the right, we see that the impact parameter b and the distances $Z_{1,2}$ are given by

$$b = \frac{rR\sin\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}}$$

$$Z_2 = R\,\frac{R - r\cos\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}}$$

$$Z_1 = -r\,\frac{r - R\cos\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}}$$



where in terms of the orbital periods and time measured from opposition !‡ • the angle between the radius vectors of the planets is

$$\theta = \pi + 2\pi \left(\frac{1}{T_2} - \frac{1}{T_1} \right) t\,.$$

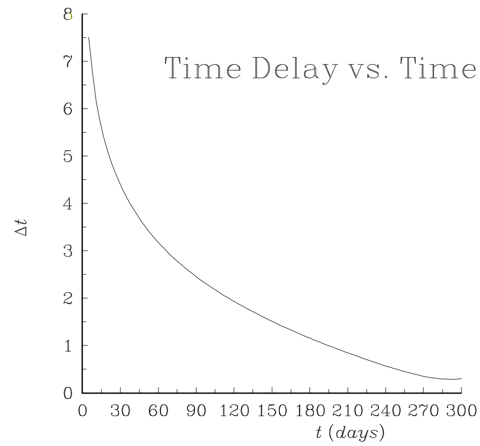
In terms of these quantities the time delay can be written

$$\Delta t \approx \frac{2M_\odot G}{c^3} \log\left(\frac{4\,|\cos\theta - r/R\,|\,1 - r/R\,\cos\theta\,|}{\sin^2\theta} \right).$$

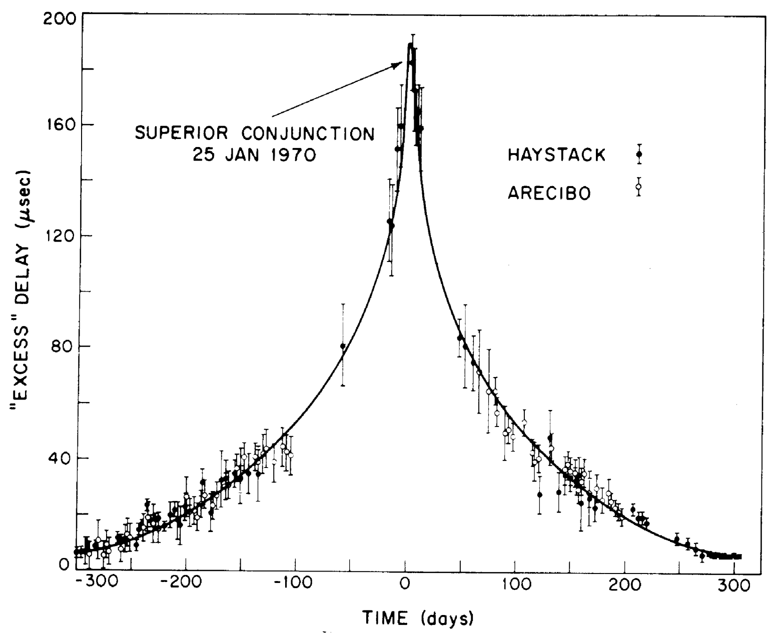
This function is plotted on the next page for the Earth-Venus measurement, in units of $\dfrac{2M_\odot G}{c^3}$.

† Note we have added the explicit factors of c required for dimensional consistency.

‡ When the Earth and Venus are on opposite sides of the Sun it is called a "superior conjunction" for reasons that go deep in the history of astronomy.



Measurements by Shapiro, *et al.*† of time delays in radar echos from the planet Venus are plotted below together with the theoretical curve(s) derived above.



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† I.I. Shapiro, *et al., Phys. Rev. Lett.* **26** (1971) 1132.