Linear field approximation, IV

Retardation of light in a gravitational field
We saw in §12 that the action for a particle in a weak field was
\[ A = \int d\tau \Lambda (\xi, \partial_{\mu} \xi) = \int d\tau \frac{1}{2} m \left( \eta^{\mu\nu} + h^{\mu\nu} \right) U_{\mu} U_{\nu} \]  
(13.1)
where \( U_{\mu} = \frac{d\xi_{\mu}}{d\tau} \), so the canonical momentum is
\[ p^{\mu} = \frac{\partial A}{\partial U_{\mu}} = m \left( U^{\mu} + h^{\mu\nu} U_{\nu} \right) \]  
(13.2)
and the equations of motion are
\[ \frac{dp^{\mu}}{d\tau} = \frac{1}{2} m U_{\kappa} U_{\nu} \frac{\partial h^{\kappa\nu}}{\partial \xi_{\mu}} = \frac{1}{2} p_{\kappa} U_{\nu} \frac{\partial h^{\kappa\nu}}{\partial \xi_{\mu}} , \]  
(13.3)
where eq. 13.3 follows from our agreement to ignore terms of higher order in \( h \). Thus for a particle travelling at near-light speed (or at \( c \), for that matter) and weakly deflected by a gravitational field, recalling that for a static source
\[ h^{\mu\nu} = \begin{pmatrix} -2MG/r & 0 & 0 & 0 \\ 0 & -2MG/r & 0 & 0 \\ 0 & 0 & -2MG/r & 0 \\ 0 & 0 & 0 & -2MG/r \end{pmatrix} \]
we have
\[ dp_{z}^{2} = -\frac{1}{2} p^{0} dt \frac{\partial h^{00}}{\partial z} - \frac{1}{2} p^{2} dz \frac{\partial h^{00}}{\partial z} \]  
(13.4)
\[ dp^{0} = 0 \]  
(static source, \( \partial_{t} h^{00} = 0 \)). Since \( dt \approx dz \) for \( v = c \), we have
\[ dp_{z}^{2} = -\frac{\partial h^{00}}{\partial z} p^{0} dz \]  
(13.5)
We may integrate w.r.t. \( z \) to get
\[ p^{2}(z) = p^{2}(\pm \infty) - p^{0} h^{00}(r) . \]  
(13.6)
The momentum at \( z = \pm \infty \) is \( \approx p^{0} \), hence we may say the group velocity of the particle at position \( z \) is
\[ v(z) \stackrel{df}{=} \frac{dp^{0}}{dp_{z}^{2}} = \left( 1 + \frac{2MG}{r} \right)^{-1} . \]  
(13.7)
The time delay in passing the object at impact parameter $b$ is then given by the integral of the time to go a distance $dz$, relative to what it would have been with no source of gravitation:

$$
\Delta t = \int_{Z_1}^{Z_2} \left( \frac{dz}{v(b, z)} - \frac{dz}{1} \right).
$$

The integral 13.8 can be performed in closed form, giving!†•

$$
\Delta t = \frac{2M_\odot G}{c^3} \left[ \sinh^{-1} \left( \frac{Z_2}{b} \right) - \sinh^{-1} \left( \frac{Z_1}{b} \right) \right].
$$

If $|Z_{1,2}| \gg b$, then the above expression simplifies to

$$
\Delta t \approx \frac{2M_\odot G}{c^3} \log \left( \frac{4|Z_2|}{|Z_1|} \right).
$$

With Earth and Venus shown as in the drawing to the right, we see that the impact parameter $b$ and the distances $Z_{1,2}$ are given by

$$
b = \frac{rR \sin \theta}{\sqrt{R^2 + r^2 - 2rR \cos \theta}}
$$

$$
Z_2 = R \frac{R - r \cos \theta}{\sqrt{R^2 + r^2 - 2rR \cos \theta}}
$$

$$
Z_1 = -r \frac{r - R \cos \theta}{\sqrt{R^2 + r^2 - 2rR \cos \theta}}
$$

where in terms of the orbital periods and time measured from opposition!‡ • the angle between the radius vectors of the planets is

$$
\theta = \pi + 2\pi \left( \frac{1}{T_2} - \frac{1}{T_1} \right) t.
$$

In terms of these quantities the time delay can be written

$$
\Delta t \approx \frac{2M_\odot G}{c^3} \log \left( \frac{4|\cos \theta - \frac{1}{2R} |1 - \frac{1}{2R} \cos \theta|}{\sin^2 \theta} \right).
$$

This function is plotted on the next page for the Earth-Venus measurement, in units of $\frac{2M_\odot G}{c^3}$.

† Note we have added the explicit factors of $c$ required for dimensional consistency.

‡ When the Earth and Venus are on opposite sides of the Sun it is called a “superior conjunction” for reasons that go deep in the history of astronomy.
Measurements by Shapiro, et al.† of time delays in radar echos from the planet Venus are plotted below together with the theoretical curve(s) derived above.