Parallel Displacement of a Vector

Reading: Mathews & Walker, *Mathematical Methods of Physics*, ch. 15.
S. Weinberg, *Gravitation and Cosmology*, ch. 3 & 4.
Ohanian & Ruffini, *Gravitation and Spacetime*, ch. 5 & 6.

Freely falling test object

We saw in §16 that the Christoffel symbol

$$\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix} \overset{df}{=} g^{\lambda\sigma}\,[\mu\nu,\sigma] = \tfrac{1}{2}g^{\lambda\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right] \tag{17.1}$$

appears in the covariant derivative of a vector

$$A_{\mu\,;\,\nu} = A_{\mu\,,\,\nu} - \begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix} A_\lambda\,. \tag{17.2}$$

We remark that we can determine the transformation law of $\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}$ from that for $A_{\mu\,,\,\nu}$: from Eq. 16.8, we have

$$\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}' = \begin{Bmatrix}\kappa\\ \sigma\alpha\end{Bmatrix}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}}\cdot\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\cdot\frac{\partial x^{\alpha}}{\partial x'^{\nu}} - \left(\frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma}\,x^{\alpha}}\right)\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\cdot\frac{\partial x^{\alpha}}{\partial x'^{\nu}}\,. \tag{17.3}$$

We now look at the equations of motion for a freely falling "test" object, which we expect to obey the variational principle

$$\delta\int d\tau = \delta\int_a^b dp\left(g_{\mu\nu}\frac{dx^\mu}{dp}\cdot\frac{dx^\nu}{dp}\right) = 0\,, \tag{17.4}$$

where p is some arbitrary parameterization of the space-time curve $x^\mu(p)$. As we have already seen, masses generate gravitational fields, which we have agreed to subsume into changes of the metric tensor (from the Minkowski tensor of flat space). A test body is thus one whose effect on the metric can be neglected.

We vary by adding to $x^\mu(p)$ an arbitrary small displacement $\zeta^\mu(p)$ that vanishes at $p = a$ and at $p = b$. Thus,

$$\tfrac{1}{2}\int_a^b dp\left(g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{-1/2}\left[2g_{\mu\nu}\frac{dx^\mu}{dp}\frac{d\zeta^\nu}{dp} + \partial_\lambda g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\zeta^\lambda\right] = 0\,. \tag{17.5}$$

But

$$\left(g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{-1/2} \equiv \frac{dp}{d\tau}\,,$$

hence change variables from p to τ, and write

$$0 = \frac{1}{2}\int_a^b d\tau \left[2g_{\mu\nu}\frac{dx^\mu}{dp}\frac{d\zeta^\nu}{dp} + \partial_\lambda g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\zeta^\lambda\right]$$
$$= -\int_a^b d\tau \left[\frac{d}{d\tau}\left(g_{\mu\lambda}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\partial_\lambda g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]\zeta^\lambda \;. \tag{17.6}$$

Thus,

$$g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \left(g_{\mu\sigma,\nu} - \frac{1}{2}g_{\mu\nu,\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

or, on multiplying through by $g^{\lambda\sigma}$, we find

$$\frac{d^2x^\lambda}{d\tau^2} + g^{\lambda\sigma}\left(g_{\mu\sigma,\nu} - \frac{1}{2}g_{\mu\nu,\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0\,.$$

If we now note that

$$g_{\mu\sigma,\nu} \equiv \frac{1}{2}\left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu}\right) + \frac{1}{2}\left(g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}\right)$$

and that the piece that is antisymmetric in $\mu\nu$,

$$\frac{1}{2}\left(g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}\right),$$

vanishes when contracted with the (symmetric) factor $\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$, then, clearly,

$$\frac{d^2x^\lambda}{d\tau^2} + \begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0\,. \tag{17.7}$$

This is the equation describing the behavior of a falling body. It agrees, to $O(h^2)$, with our previous resuult for the linearized gravitaional field, Eq. 11.14.

Parallel displacement of a vector along a curve

The equation 17.7 of a freely falling test body now brings us to the question of *parallel displacement of a vector*.

We see that in a system of coordinates ξ^α falling freely with the test object (so-called *co-moving coordinates*) the acceleration vanishes,

$$\frac{d^2\xi^\lambda}{d\tau^2} = 0\,,$$

hence the Christoffel symbol $\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}$ vanishes also.

The significance of freely falling coordinates is that, in the neighborhood of a point we can consider the space free of gravitational effects†. Since the derivatives of the metric tensor with respect to coordinates can be expressed as Christoffel symbols, the metric tensor can obviously be considered Minkowskian in such a system, up to terms in its second derivatives:

$$g_{\mu\nu\,,\,\sigma} = \left[\mu\sigma, \nu\right] + \left[\nu\sigma, \mu\right] = 0\,,$$

$$\therefore\;\; g_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{2}\frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\,\partial x^\lambda}\,\xi^\kappa\,\xi^\lambda\;.$$

That is, the space is *locally flat*, and deviations from flatness are quadratic in the coordinates.

Now consider a vector $\widetilde{V}^\alpha$ that is *constant* with respect to a freely falling system of coordinates ξ^α; and a space curve $x^\mu\,(p)$ (parameterized by an invariant parameter *p*----such as τ). Because

$$\frac{d\widetilde{V}^\alpha}{dp} = 0\,,$$

we can calculate the derivative of

$$V^\mu = \frac{\partial x^\mu}{\partial \xi^\alpha}\,\widetilde{V}^\alpha \tag{17.8}$$

with respect to *p*, *i.e.* along the curve, in an arbitrary coordinate system:

$$\frac{dV^\mu}{dp} = \left(\frac{\partial^2 x^\mu}{\partial\xi^\alpha\,\partial\xi^\beta}\right)\frac{\partial\xi^\beta}{dp}\,\widetilde{V}^\alpha = \left(\frac{\partial^2 x^\mu}{\partial\xi^\alpha\,\partial\xi^\beta}\right)\frac{\partial\xi^\alpha}{\partial x^\sigma}\,\frac{\partial\xi^\beta}{\partial x^\lambda}\,\frac{dx^\lambda}{dp}\,V^\sigma$$

$$\equiv -\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}\frac{dx^\lambda}{dp}\,V^\sigma\;. \tag{17.9}$$

That is, the condition that *V* be "constant" when transported from one point to another along a curve, with respect to a "flat" space, is that

$$\frac{dV^\mu}{dp} + \begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}V^\sigma\,\frac{dx^\lambda}{dp} = 0\,. \tag{17.10}$$

Equation 17.10 is sometimes called the *equation of parallel transport*. Given a vector field $V^\sigma\,(x)$ whose value at the point a^μ is $V^\sigma\,(a)$, and a space-time curve $x^\mu\,(p)$ joining the point $a^\mu = x^\mu\,(p)$ with another point $b^\mu = x^\mu\,(p{+}dp)$, we can construct a 4-tuple $\overline{V}^\sigma\,(p{+}dp)$----defined with respect to the (different) coordinate system at b^μ----that is parallel to the first in the above sense, *via*

$$\overline{V}^\sigma\,(p{+}dp) \overset{df}{=} \overline{V}^\sigma\,(p) - \begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}V^\sigma\,(a)\,\frac{dx^\lambda}{dp}\,dp\,, \tag{17.11}$$

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† This is where the Principle of Equivalence comes in, since gravitation and acceleration are evidently indistinguishable at a point.

where, by definition, $\overline{V}^{\mu}(p) \stackrel{df}{=} V^{\mu}(a)$. The new 4-tuple is indeed a vector:

$$\frac{d}{dp}\left[\overline{V}'^{\,\mu}(p) - \frac{\partial x'^{\,\mu}}{\partial x^{\sigma}}\,\overline{V}^{\sigma}(p)\}\right] = 0\,. \qquad (17.12)$$

Problem for the fearless:

Show Eq. 17.12 is correct. (Hint: use Eq. 17.3.)

Thus,

$$\overline{V}'^{\,\mu}(p) - \frac{\partial x'^{\,\mu}}{\partial x^{\sigma}}\,\overline{V}^{\sigma}(p) = \text{constant}\,.$$

At the point a^{β}

$$\overline{V}'^{\,\mu} = \frac{\partial x'^{\,\mu}}{\partial x^{\sigma}}\,V^{\sigma}(a)$$

hence the constant is zero!